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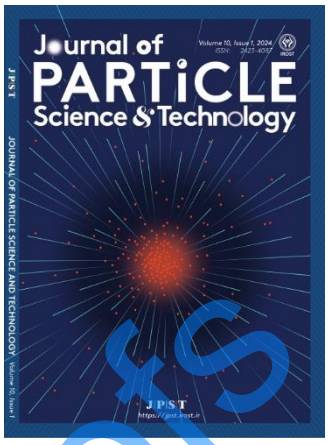

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Arezu Jahanshir

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Pionic atom localized states under the relativistic effect and memory model

Arezu Jahanshir

Department of Physics and Engineering Sciences, Buein Zahra Technical University, Iran

Abstract

The presented work is innovative in proposing a novel equation for the velocity of constituent particles in relativistic localized states, derived from a unique synthesis of quantum field theory, memory models, and harmonic-oscillator dynamics. This approach offers a new way to model the internal dynamics of hadronic systems, with potential applications in nuclear physics and exotic atoms. Its primary strength lies in integrating disparate theoretical frameworks to describe complex quantum phenomena effectively. We introduce a significant physical equation for calculating the velocity of constituent particles in a localized state, derived from quantum field theory, the relativistic behavior of interactions, and a memory model for moving mass. This equation is obtained from the ratio of the relativistic mass to the rest mass of the constituent particles, together with Boltzmann's constitutive equations for a quantum harmonic-oscillator system that incorporates the Kelvin–Voigt memory model and the features of quantum field theory characteristic of the strong interaction within a hadronic localized state. Initially, we determine the relativistic masses of the constituent particles in pionic-atom localized states (bound states) by solving a modified Schrödinger equation in a symplectic space constructed from two intertwined subspaces generated by the creation and annihilation operators. We then employ the Kelvin-Voigt memory model equation to determine the velocities of the particles in the relativistic regime within the framework of the strong interaction among hadrons. The theoretical predictions for the pionic atom are in good agreement with current experimental data.

Keywords: Bound state; Exotic atom; Schrödinger equation; Kelvin-Voigt memory model; Path integral

1. Introduction

The fundamental behaviors of interacting particles at high energies with the large coupling constant of interaction are based on the relativistic and quantum field theory (QFT) mechanism of interactions. In addition to how quantum particles interact with each other in different potential models, the mass and speed of particles also play an important role in the hadronic interaction process. A two-particle system in a localized state, within the framework of the strong force and quantum chromodynamics theory, can serve as a good model for this study. This model allows us to describe the hadronic objects' interaction at short distances, as well as exotic atoms and exotic localized states [1]. Accordingly, we consider exotic hadronic localized states such as the pionic atom (π-atom or pionium) in the relativistic limit under the strong Coulombic interaction [2,3]. The constant of the attractive Coulombic interaction between π^+ and π^- is large, with a coupling constant $\alpha_s: 0.1 - 0.5$ in the strong-coupling regime. Within relativistic QFT, the main features of a pionic atom are described by the transition amplitude and the orbital quantum number, which are determined from an integral equation of either the Schrödinger or Bethe–Salpeter equation types. It is widely recognized that, within conventional QFT, the nonperturbative character of the attractive Coulombic interaction between π^+ and π^- deals with solving an integral equation with an arbitrary kernel, a task that is generally difficult. For a large coupling constant, the different types of interaction diagrams do not affect the real character of π^+ and π^- interaction; hence, nonperturbative and relativistic theory can be used to obtain the characteristics and properties of exotic pionic atom states in the range $\alpha_s: 0.1 - 0.5$ [3]. The Feynman functional path integral and the Feynman-Schwinger representation (FSR) of quantum chromodynamics are among the most powerful theoretical and mathematical methods for describing quantum systems governed by the strong interaction at large distances, linking the nonperturbative dynamics of scalar field theory to the relativistic framework of π^+ and π^- interaction in its localized state [4]. These frameworks provide a powerful tool and an alternative formulation for calculating scattering amplitudes, the output of mass-spectrometry output, and the effective quark

mass (constituent mass) in the localized (bounded) state, as well as decay rates in quantum field theory. This approach can be applied to represent the statistical correlation function (the polarization Green's function (GF)) as a path integral, which allows us to solve the integral equation of the attractive Coulombic interaction between π^+ and π^- under certain specific physical assumptions, for calculating the observables in QFT at large distances, based on the nonperturbative character of the interaction [3,5]. After defining the relativistic mass of particles using the Kelvin-Voigt memory model, which was first presented by Boltzmann [6], we define the relativistic speed of the π^+-π^- pion and of a pionic atom in the strong interaction, at a coupling constant α_s in the range $0.1 - 0.5$. Boltzmann proposed that the linear relationship between the load $F(t)$ (an external force or external field) and the elastic load-deflection $\xi(t)$ depends on a third parameter, time t. Thus, the memory elastic load-deflection $\xi(t)$ for an object of mass m is proportional to the external force $F(t)$, with a constant of proportionality $\beta(t)$ [7-12]. The paper is organized as follows: in Section 2, we present the representation of the polarization GF in the FSR form, as constructed within QFT, to define the pionic atom mass spectrum of the localized state and to present the effective quark mass formula for π^+ and π^- in these localized states. In section 3, we describe the constitutive equations of a localized state that depend on two or more physical quantities, specific to the mass, speed, and other observables, which approximate its response to an external force or applied field. These fundamental calculations are combined with other equations and physical laws governing the underlying physics to solve the binding-particle problems in solid state physics and to relate applied tensions and loads to strains and/or deformations and analogous viscoelastic models employed in nanomechanics [13,14]. We then define the connection between the deformation principle in structural mechanics and in particle physics. In section 4, we describe the memory model of moving particles as used in structural-mechanical-engineering concepts and then extend this model to particle fields. In Section 5, we construct a generalized description of the localized state under the memory model in the relativistic limit, at large distances and high energy levels, and obtain theoretical results for the

relativistic speed of the pionic atom and for π^+ and π^- in these localized states, based on the QFT and memory model approaches, are presented. Finally, a summary and conclusion are given.

2. Mass spectrum in the Quantum Field Theory

One of the main approaches for determining the localized state mass spectrum in particle physics is based on the relativistic behaviors of the interaction and on quantum field theory. In this study, we consider the interaction of two charged particles at positions $\mathrm{r}_1(\tau_1), \mathrm{r}_2(\tau_1)$, respectively, which forms a localized state at high energy levels in an external scalar field [15-19]. Thus, to define the relativistic mass of the system and the effective quark mass in the localized state, the modified Schrödinger equation (MSE), in the form of the mass-spectrum $M(\mu)$ equation (1) and the energy-eigenvalue $E_\ell(\mu)$, equation (2), in Cartesian coordinate, is considered as follows:

$$\widehat{H}\Psi(\mathrm{r}) = M(\mu)\Psi(\mathrm{r}) \tag{1}$$

$$\widehat{H}\Psi(\mathrm{r}) = E_\ell(\mu)\Psi(\mathrm{r}) \tag{2}$$

where the parameter μ is the concentrated mass (reduced mass) of the system, $\mathrm{r} = \mathrm{r}_1(\tau_1) - \mathrm{r}_2(\tau_2)$, and $\tau = \tau_1 - \tau_2$, τ_1, τ_2 are the proper times of the components with masses m_1, m_2. As is well known, in QFT [16-21], the mass of a scalar particle is presented by the Gaussian statistical correlated function $\Pi(\mathrm{r})$, and the total orbital quantum numbers ℓ using the real current density of charged scalar particles associated with the flow of particles at a specific position $\vec{\mathrm{r}}$ in the spacetime coordinates, based on the field operators' generation $\Phi^+(\mathrm{r})$ and destruction $\Phi^-(\mathrm{r})$ operators in the context of QFT and fermionic field theory can be formed by current $J(\mathrm{r})$: $J(\mathrm{r}) = \Phi^+(\mathrm{r})\Phi^-(\mathrm{r})$ and, by the complex conjugate field $\varphi(\mathrm{r})$ as $\mathrm{J}(\mathrm{r}) = \frac{\mathrm{i}\hbar}{2\mathrm{m}}\left(\varphi\frac{\partial}{\partial \mathrm{r}_\mu}\varphi^* - \varphi^*\frac{\partial}{\partial \mathrm{r}_\mu}\varphi\right)$, where $\frac{\partial}{\partial \mathrm{r}_\mu}$ is the four-gradient $\left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$. The correlation function in $G\left(\mathrm{r}\middle|A_\mu\right)$ concepts read [13,14-19]

$$\left[\left(i\frac{\partial}{\partial \mathrm{r}_\mu}+\frac{g}{c\hbar}A_\mu\,(\mathrm{r})\right)^2+\frac{c^2m^2}{\hbar^2}\right]G\left(\mathrm{r}\middle|A_\mu\right)=\delta(\mathrm{r}) \tag{3}$$

where m is the mass of the scalar particles, g is the interaction coefficient, $A_\mu\,(\mathrm{r})$ is the external gauge field, and δ is Dirac's delta function. If we disregard the destruction channels, the correlators are defined by the product of the scalar charged field GFs. Therefore, averaging over the field $A_\mu\,(\mathrm{r})$ by the relation

$$\langle exp([i\int d\,\mathrm{r}_1\,J_\mu\,(\mathrm{r}_1)A_\mu])\rangle = exp(\tfrac{-1}{2}[\int\int d\mathrm{r}_1\,d\mathrm{r}_2\;J_\mu\,(\mathrm{r}_1)G_{\mu\nu}(\mathrm{r}_1-\mathrm{r}_2)J_\nu\,(\mathrm{r}_2)]) \tag{4}$$

the propagator function $G_{\mu\nu}(\mathrm{r}_1-\mathrm{r}_2)$ of the external field $A_\mu\,(\mathrm{r})$ is defined by the relation

$$G_{\mu\nu}\big((\mathrm{r}_1-\mathrm{r}_2)\big|A\big)=\langle A_\mu(\mathrm{r}_1)A_\nu(\mathrm{r}_2)\rangle=\delta_{\mu\nu}G(\mathrm{r}_1-\mathrm{r}_2)+\partial_\mu\partial_\nu G_1(\mathrm{r}_1-\mathrm{r}_2) \tag{5}$$

and then [20-24]

$$G_{\mu\nu}\left((\mathrm{r}_1-\mathrm{r}_2)\middle|A_\mu\right)=\int\left(\frac{dk}{2\pi}\right)^4\tilde{G}(k^2)\times exp(ik(\mathrm{r}_1-\mathrm{r}_2)) \tag{6}$$

We can fix the gauge of the external field, since the physical properties of the system do not depend on $\tilde{G}(k^2)$. The propagator function $G_{\mu\nu}(\mathrm{r}_1-\mathrm{r}_2)$ of the strong Coulombic interaction reads [16]:

$$G_{\mu\nu}\left((\mathrm{r}_1-\mathrm{r}_2)\middle|A_\mu\right)=\frac{1}{(\mathrm{r}_1-\mathrm{r}_2)^2(2\pi)^2} \tag{7}$$

This function defines the properties of the strong Coulombic interaction in the external field A_μ. As is generally acknowledged, in quantum field theory, the localized states refer to the stable or pseudo-stable binding of particles within the strong interaction, which is strong enough to keep the particles together, despite the repulsive potential. A localized state is characterized by the constituent particles having a finite probability of being found within a certain region of the spacetime coordinates. When we refer to a localized state's behavior at large distances, we mean that the particles remain connected and correlated with each other even at significant separation distances, i.e., in long-range interactions. Localized states at large distances arise across various fields of

physics, such as QFT, particle physics, and high-energy physics. This has led us to study complex hadronic systems and their phenomena. Accordingly, the localized state mass spectrum of two scalar particles in the external field A_μ is constructed from the asymptotic behavior of the statistical correlation function $\Pi(\mathrm{r}_1-\mathrm{r}_2)$ for two charged scalar particles with the rest masses m_1, m_2 in an external field A_μ, which has the form:

$$\Pi(\mathrm{r}_1-\mathrm{r}_2)=\langle G_{m_1}(\mathrm{r}_1,\mathrm{r}_2|A)\cdot G_{m_2}(\mathrm{r}_1,\mathrm{r}_2|A)\rangle_A \approx exp(-M|\mathrm{r}_1-\mathrm{r}_2|) \tag{8}$$

Equation (8) may be interpreted as the Euclidean two-point correlation function (propagator) of a composite operator. At large separations, $|\mathrm{r}_1-\mathrm{r}_2|\to\infty$, the correlator is dominated by the lowest-lying state coupled to the operator and therefore exhibits the asymptotic behavior $\approx e^{-M|\mathrm{r}_1-\mathrm{r}_2|}$, where M is the mass of the corresponding localized two-particle state. The exponential decay of the correlator thus provides direct access to the mass spectrum of the theory and encodes the effects of quantum fluctuations and interactions entering through the propagator, and then reads:

$$M=\left[-\lim \frac{1}{|\mathrm{r}_1-\mathrm{r}_2|} ln\Pi(\mathrm{r}_1-\mathrm{r}_2)\right]_{|\mathrm{r}_1-\mathrm{r}_2|\to\infty} \tag{9}$$

Equation (9) defines the mass spectrum of the localized states if $M\neq m_1+m_2$, $M\prec\infty$. Hence, we must first solve equation (3), as we have already outlined the method for solving this type of equation in Section 1. We can now present the solution of equation (3) using the functional integral approaches, as supported by the FSR method [15], in the following form ($\hbar=c=1$)

$$G(\mathrm{r}_i|A_\mu)=\int_0^\infty d\mu_i\,\delta(\mathrm{r}_1-\mathrm{r}_2)\times exp\left(-m_i^2\mu_i-\int_0^1 d\tau\left(\frac{\mu_i i\,\partial}{\partial \mathrm{r}_\mu}+\mu_i g A_\mu(\mathrm{r}_i(\tau))\right)^2\right) \tag{10}$$

and then leads

$$G(\mathrm{r}_i|A_\mu)=\int_0^\infty d\beta\,\delta(r))\;\; exp\left[-m_i^2\mu_i-T_\tau\beta\int_0^1 d\tau\left(\mu_i i\frac{\partial}{\partial \mathrm{r}_\mu}+\mu_i g A_\mu(\mathrm{r}_i(\tau))\right)^2\right] \tag{11}$$

where T_τ is the chronological-ordering operator. Using the propagator function in the form of (11)

and defining by new variables, after some calculation and averaging over the field just two Gaussian correlators $\langle e^{[i\int dx A_\mu(\mathrm{r}_i)J(\mathrm{r}_i)]}\rangle_{A_\mu} = e^{\left[-0.5\int\int d\mathrm{r}_i\, d\mathrm{r}_j\, J(\mathrm{r}_i)D(\mathrm{r}_i;\mathrm{r}_j)J(\mathrm{r}_j)\right]}$, where the propagator of the gauge field $D\left(\mathrm{r}_i;\mathrm{r}_j\right)$ reads $D\left(\mathrm{r}_i;\mathrm{r}_j\right) = \langle A_\mu(\mathrm{r}_j)A_\mu(\mathrm{r}_i)\rangle_{A_\mu} = \left[\partial^2\mathfrak{D}_1(\mathrm{r}_i;\mathrm{r}_j) + \delta\breve{D}(\mathrm{r}_i;\mathrm{r}_j)\right]$. Then We can define the asymptotic behavior $|\mathrm{r}_1 - \mathrm{r}_2| \to \infty$ of the polarization operator $\Pi(\mathrm{r}) = \langle G_1; G_2\rangle_A$ of the interaction between particles with mass m_1 and m_2, that forms a localized bound state in the external gauge field A_μ and has a nonlocal interaction $J_\mu(\mu_1,\mu_2)$is defined. After simplifying equations (9) and (10), the statistical correlation function of the mass spectrum reads (see [11] for full details):

$$\Pi(\mathrm{r}) = \int_0^\infty\int_0^\infty \frac{d\mu_1 d\mu_2}{(8\mathrm{r}\pi^2)^2}\; J_\mu(\mu_1,\mu_2)\; exp\left[-\frac{\mathrm{r}}{2}\left(\left[\frac{m_1^2}{\mu_1}+\mu_1\right]+\left[\frac{m_2^2}{\mu_2}+\mu_2\right]\right)\right] \tag{12}$$

and

$$J_\mu(\mu_1,\mu_2) = N_1N_2\iint \delta r_1\delta r_2\; exp\left[-V_{1,1}+V_{1,2}-V_{2,2}\right] exp\left[-\frac{1}{2}\left(\int_0^{\mathrm{r}} d\tau\left(\mu_1\dot{r}_1^2(\tau)+\mu_2\dot{r}_2^2(\tau)\right)\right)\right] \tag{13}$$

The functional integral $J_\mu(\mu_1,\mu_2)$ is evaluated under the FSR [18] formalism in nonrelativistic quantum mechanics for R^4 dimensional motion of particles with velocities $v_1(\tau), v_2(\tau)$, potential and non-potential interactions $V_{i,j}$ of particles with masses μ_1,μ_2. According to the above, the energy eigenvalue of the localized states of two particles with masses m_1, m_2 appears to be related to the correlation function $J_\mu(\mu_1,\mu_2)$ near $|\mathrm{r}_1 - \mathrm{r}_2| \to \infty$ and reads

$$J_\mu(\mu_1,\mu_2) \approx \exp(-|\mathrm{r}_1 - \mathrm{r}_2|E_\ell(\mu_1,\mu_2)) \tag{14}$$

Using the extension of Laplace's method for approximating the integral $I(\lambda) = \int_C dz f(z)\exp(\lambda g(z))$, and the method of steepest descent, we can obtain the approximation value of the above integrals. Thus, in the large-scale behavior of the polarization function $\Pi(\mathrm{r})$ near $|\mathrm{r}| \to \infty$, the mass spectrum of the localized state, using the saddle-point method, reads:

$$M = \min_{\mu_1,\mu_2}\left(\frac{m_1^2\mu_2+\mu_1m_2^2}{2\mu_1\mu_2}+\frac{\mu_1+\mu_2}{2}+E(\mu_1,\mu_2)\right) \tag{15}$$

where $E(\mu_1,\mu_2)$ is the energy eigenvalue of the localized state of the SE (2) in the nonrelativistic limit for the potential interaction, and μ_1,μ_2 is the effective quark mass inside the localized state, i.e., the relativistic mass of the particles with relativistic velocities $v_1(\tau), v_2(\tau)$, and rest masses m_1, m_2. Here, according to the SE for the localized state, we can introduce the parameter μ. This is the relativistic reduced mass of the system and is defined by the equation $\frac{1}{\mu}=\frac{1}{\mu_1}+\frac{1}{\mu_2}$. We therefore use the relativistic energy approximation :

$$\sqrt{m^2+\hat{p}^2}\approx\min_{\mu}\left(\mu+\frac{m^2+\hat{p}^2}{\mu}\right) \tag{16}$$

and, originating from the relativistic energy in the SE $\hat{H}\Psi(\mathrm{r})=E_\ell(\mu)\Psi(\mathrm{r})$, in the following form [13,25]:

$$\left[\sqrt{m_1^2+\hat{p}^2}+\sqrt{m_2^2+\hat{p}^2}+V(r)\right]\Psi(\mathrm{r})=0 \tag{17}$$
$$-E_\ell(\mu)$$

This approximation method enables us to solve for and predict the structure of the localized states using the SE, in the natural unit ($\hbar=c=1$): $\hat{H}\Psi(\mathrm{r})=E_\ell(\mu)\Psi(\mathrm{r})$. Hence, the MSE $\hat{H}R(r)=E_\ell(\mu_1,\mu_2)R(r)$ for a quantum mechanical system, reads [15]:

$$\left[\frac{1}{2}\left(\mu_1+\frac{m_1^2+\hat{p}_r^2}{\mu_1}\right)+\frac{1}{2}\left(\mu_2+\frac{m_2^2+\hat{p}_r^2}{\mu_2}\right)+V(r)-E_\ell(\mu_1,\mu_2)\right]R(r)=0 \tag{18}$$

For a pionic bound state which, we describe completely the wavefunction $R(r)$of Schrödinger equation in (18) the standard and physically appropriates boundary conditions in the form of 1) at the origin: the reduced radial wavefunction must be regular, $\mathrm{R}_{n\ell}(\mathrm{r})\propto \mathrm{r}^{\ell+1}, \mathrm{r}\to 0$, and 2) at infinity: the bound-state wavefunction must be normalizable, $\mathrm{R}_{n\ell}(\mathrm{r})\to 0, \mathrm{r}\to\infty$, with exponential decay. For the phenomenological Kelvin–Voigt equations, the usual initial conditions specifying the state of the pionic bound state at the interaction, can consider $m(0)=m_0, v(0)=0$.These conditions are the most natural and consistent with the physical interpretation of this theoretical research. The localized mass M of the system and the constituent masses of its components μ_1,μ_2 are determined by the

steepest-descent method and are defined by equations

$$M = \min_{\mu_i,\mu_j}\left(\frac{m_i^2}{2\mu_i}+\frac{m_j^2}{2\mu_j}+\frac{\mu_i+\mu_j}{2}+E_\ell\right) \tag{19}$$

where the constituent mass $\mu_i$of each particle in the bound state, that is, the relativistic mass of the them, inside the pionium within the interaction potential, is determined by minimizing (19) in the form of $2\mu_i^2\frac{dE(\mu_i)}{d\mu_i}+\mu_i^2-m_i^2=0$, and defined by (20).

$$\mu_i^2 = m_i^2 - 2\mu_i^2\left(\min_{\mu_i} E_\ell\right) \tag{20}$$

This form of equation (18) is usually referred to as the relativistic SE and can be approximately equivalent to the Bethe-Salpeter equation or the Dirac equation if spin interactions are included.

3. The Memory Model of Moving Particle

A context of deformation, deflection, elastic load-deflection, stress, and strain, which describes the behavior of matter, can be introduced in structural and continuum mechanics. There are specific conditions under which each of these parameters exists. In continuum mechanics, load-deflection $\xi(t)$ is a physical quantity that shows how much a physical system will deform when subjected to an external load $F(t)$ over the duration t of the external force's effect i.e., when an external force $F(t)$ is applied to a system over that time interval t, it will experience a certain amount of deflection $\xi(t)$ over that time interval t, which may cause it to stretch, compress, bend or twist [2,7,28]. We consider the load deflection to exhibit linear or non-linear behavior under a varying force over a time interval $dt = t_2 - t_1$. Thus, one can express the velocity of the deflection over a time interval $\frac{dx(t)}{dt}$, which is named the elastic load-deflection rate, in a one-dimensional coordinate system along the x-axis $\xi(t) = \frac{dx(t)}{dt}$. The variation of $\xi(t)$ over the time duration, dt is directly proportional to the first-order term of applied load or applied external force $F(t)$ over the duration dt, i.e., $\xi(t) = \beta(t)\dot{F}(t)$. Here, the characteristic function of the system (or matter) is introduced as a function $\beta(t)$; it is the

proportionality constant, which is assumed to be a constant function of time. The load-deflection $\xi(t)$ can take an elastic or non-elastic form. Elastic load-deflection refers to a temporary deformation, i.e., when the external force is removed, the matter or structure is fully recovered, as long as it remains within its elastic limits. The elastic load-deflection is called "linear" if the relationship between $\xi(t)$ and $F(t)$ is directly proportional and follows Hooke's Law ($F(t) = kx(t)$). Otherwise, it is referred to as a non-linear elastic load-deflection [28-31]. Based on the elastic load-deflection, the characteristic function can determine the frequency-dependent response, the time-domain response, and the viscoelastic properties of the system. Thus, one can define this mathematically as $\xi(t) = \beta(t)\dot{F}(t)$. According to the boost in the applied external force within the physical system (or mass) in the direction of the chosen material axis, over a small-time interval $\Delta t = t_2 - t_1$. The variation of the elastic load-deflection is defined as follows:

$$\frac{d}{dt}\left(\frac{\Delta x(t)}{\Delta t}\right)_{\Delta t\to 0} = \beta(t)\dot{F}(t) \Rightarrow \xi(t) = \int_0^t \left[\beta(t)\dot{F}(t)\right] dt \tag{21}$$

Now, we describe a system with memory as another important class of material, or mass with distinctive features and characteristics. The theoretical and mathematical formulation of the memory model principles and methods was initially described and determined by Boltzmann [1,28]. In continuum physics, memory refers to a specific effect that governs the behavior of a system in response to external fields, such as electromagnetic, gravitational, and thermal fields. It can be said to describe a connection between the external and internal fields. All physical equations governing the external field act in the time interval dt on the system and depend on the history of the system's states. In structural mechanics, memory concepts for materials with hereditary effects are modeled by a combination of springs and dashpots (devices for damping shock or vibration). The simplest such model is the Kelvin–Voigt model [28], formed of a parallel combination [1] of a linear spring and a dashpot Fig. 1.

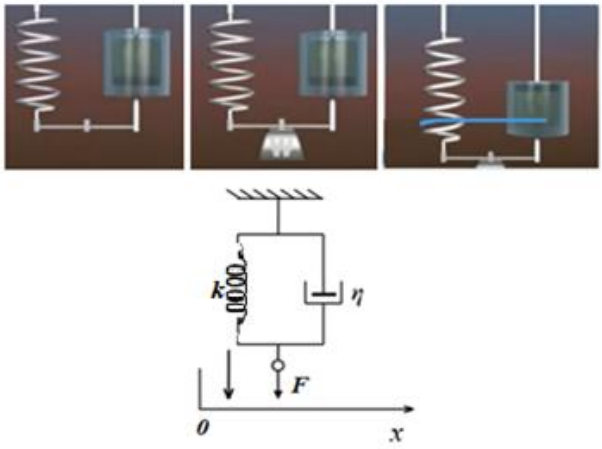

Fig. 1. The Kelvin–Voigt model of spring and dashpot.

Thus, based on the above description, the elastic load-deflection for the Kelvin–Voigt model is defined with the linear spring constant k and the damping coefficient of the dashpot η (for a solid system), as shown in Fig. 1. The spring constant determines the spring's hardness (stiffness) based on Hooke's Law, resonant frequency (of oscillation or vibration), energy storage, and the elastic limit. The damping coefficient determines the degree of damping (resistance) to motion provided by the dashpot element (the device for damping shock or vibration). Now, by suddenly applying $F(t)$ at $t = 0$, with deformation at $t = \infty$, we can consider a spring, where $F(t) = F_k(t) + F_\eta(t)$. One can see that the spring force $F_k(t)$ creates and constructs deformation, and has a linear relationship between the applied force $F_k(t)$ and the resulting displacement $x(t)$, which is determined by the relation $F_k(t) = kx(t)$. Conversely, the dashpot produces a damping force $F_\eta(t)$, which reduces the elastic load-deflection (deformation). The dashpot force or the resisting force $F_\eta(t)$, relates to the damping coefficient and the velocity of deformation $\xi(t) = \frac{dx(t)}{dt}$ by the relation $F_\eta(t) = \eta\xi(t)$. Using equation (19) for the Kelvin–Voigt model (i.e., the parallel combination of the spring and the dashpot), the total deformation force of the system (the spring force and the damping force), when they are in dynamic balance, reads

$$F(t) = F_k(t) + F_\eta(t) = kx(t) + \eta\frac{dx(t)}{dt} \tag{22}$$

and, as we know, in a parallel combination, the elongation produced by the time-varying force $F(t)$ is the same for both the spring and the dashpot $\left(\Delta x = \Delta x_k = \Delta x_\eta\right)_{\to 0}$. In the above equation, we consider the changes of forces with respect to dt and take the first derivative; equation (20) then reads

$$\dot{F}(t) = k\frac{dx(t)}{dt} + \eta\frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = k\xi(t) + \eta\dot{\xi}(t) \tag{23}$$

Equation (20) shows that $F_k(t)$ produces deformation while $F_\eta(t)$ reduces it. The simultaneous existence of $F_k(t)$ and $F_\eta(t)$ gives the system or matter its memory. In continuum mechanics, materials with memory are also referred to as materials whose constitutive equations depend on the history of thermodynamic, kinetic, electromagnetic, or other types of state variables. Thus, under the conditions: a) $\dot{\xi}(t) = constant$, b) zero initial strain, and c) constant stress σ, one defines a formula for the characteristic function $\beta(t)$ at time dt. By applying sudden force $F(t)$ at $t = 0$, we consider $F(t) = constant$, and express using the Heaviside step function $H(x)$, whose derivative is the Dirac delta function $\delta(x) = \frac{dH(x)}{dx}$, hence, equations (19) and (20) give us the load-deflection $\xi(t)$ as follows

$$\xi(t) = \frac{1}{\eta}F_\eta(t) = \frac{F(t)-F_k(t)}{\eta} = \frac{k}{\eta}\left(\frac{F(t)}{k} - x(t)\right) \tag{24}$$

where the Heaviside step function $H(x)$ and its derivative the delta Dirac function $\delta(x)$ read

$$H(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x = 0 \end{cases}, \qquad \delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{25}$$

Hence, the elastic load-deflection $\xi(t) = \frac{dx(t)}{dt}$ by applying a sudden force $F(t)$ reads

$$\frac{dx(t)}{dt} = \frac{k}{\eta}(x(t_\infty) - x(t_0)) \Rightarrow \frac{dx(t)}{(x(t_\infty) - x(t_0))} = \frac{k}{\eta}dt \tag{26}$$

After integrating equation (20) over time, the elongation $dx(t)$ and the elastic load-deflection $\xi(t)$ produced by a time-varying, suddenly applied force, where $F(t) = aH(t)$, represents the best

linear approximation of an external force that may otherwise be a non-linear function, reads

$$\left[x(t) = \frac{1}{k}F(t)\left(1 - \exp(-\frac{kt}{\eta})\right)\right]_{\Delta t} \Rightarrow \dot{\xi}(t) = \frac{1}{k}\left(1 - \exp(-\frac{kt}{\eta})\right)a\dot{H}(t) \quad (27)$$

Then the characteristic function using equation (27) and $\dot{\xi}(t) = \beta(t)a\dot{H}(t)$ defines in the following manner [28-30]

$$\beta(t) = \frac{a}{k}\left(1 - \exp(-\frac{kt}{\eta})\right) \quad (28)$$

According to the above equations and the characteristics of these two physical elements (the spring and the dashpot), the spring component captures the viscosity and elasticity behavior of the matter, while the dashpot component reduces the elastic load-deflection (deformation) and influences the overall viscosity and elasticity behavior of the matter. With this approach, the concept of deflection and deformation can be extended to a quantum-relativistic system under an external field to obtain more information about the system and its interaction within the field. We show, in this study, how the load-deflection behavior of matter is connected to the relativistic parameters and how it explains the velocity of systems such as particles and localized states in high-energy interactions. We first describe and present the mathematical formulation of the load-deflection. In general, under relativistic conditions of a system or mass, when an external field acts on a mass, the mass will experience a certain amount of spatial displacement, or change in other physical quantities that occurs due to the applied external force; and over time, this can cause change in some of the system's parameters and quantities, such as mass, velocity, and acceleration. As the external field acts on a system, the deflection counteracts the external field by absorbing it. The greater the external field (or deflection) acting on a system, the more force it can withstand, which results in an increase or decrease in the magnitude of some physical parameters and quantities in the system, without significant loss in the system's original interaction behavior. Therefore, load-deflection is a fundamental concept in physics

and is often used to analyze the system's behavior under external fields. When an external field acts on a mass (physical system or matter), it generates an internal field (or internal force) within the mass. This internal field causes the mass to deflect or deform from its original state. The relationship between the applied external field and the resulting deflection can vary depending on the mass, geometry, boundary conditions, and interaction conditions, and can be represented by a deflection curve, which shows how the physical quantities change when an external field is applied. The deflection curve can provide valuable information about the mass, the velocity limit of the mass, and the overall behavior of the quantum-relativistic system. In this study, we assume that the external field (or load-deflection) exhibits linear behavior (linear elasticity), such that the deflection is directly proportional to the applied external force. We present a theoretical formula to determine the effect of the external field within the system (or mass) and compare the results with experimental data. This theoretical formulation allows us to investigate how different interaction constants affect the deflection and to assist theoretical and experimental particle physicists in describing and providing better, more accurate observations and tracking of fundamental particle systems.

4. Relativistic Conditions Under the Memory Model

The usual Cartesian Euclidean "spacetime texture" around us refers to the patterns of spacetime coordinates that make up a *4D* continuum: x_i, t. This pseudo-Euclidean spacetime, coordinates Fig. 2, with the length dimension, is named "Minkowski coordinates". In 1908, the "Minkowski coordinate" was named after Hermann Minkowski. Using mathematical physics, he combined a *3D*: $x_i = x, y, z$ inertial space and time t coordinate with a non-inertial coordinate of space and time into a R^4 dimensional space: x_i, t model. Minkowski coordinates are directly associated with relativistic theory [21,22]. They are one of the best mathematical constructions by which Einstein's theory of relativity is described.

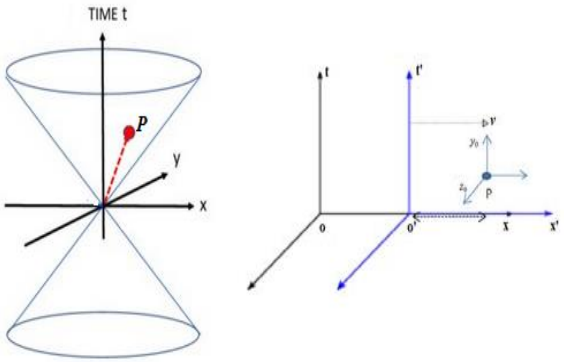

Fig. 2. Light cone in R^2 space plus the time dimension (right) and Minkowski spacetime 4D continuum (left) for an event.

For all inertial Minkowski coordinate frames (IMFs) of reference, the distance, ds, between two events in the Minkowski spacetime coordinate is defined by the Pythagorean theorem $ds = \sqrt{(d\tau)^2 + (dx_i)^2}$. Thus, in the relativistic limit, where $v \approx c$, the motion of a particle is described by a Lorentz invariance [22]. In this case, an event is restricted to moving only in one direction (x; x') in the two IMFs S and S', with the relative velocity v between them. Hence, the kinematic equations of motion based on IMFs S and S' described by "Lorentz transformations", refer only to transformations between the IMF frames, in the context of special relativity in natural unit ($\hbar = c = 1$), as follows:

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx) \tag{29}$$

where the Lorentz factor is γ. The distance ds between two points in the two-dimensional relativistic IMF continuum is defined by $ds = \sqrt{(d\tau)^2 + (dx)^2}$, which represents the time and space relations in the relativistic limit, where the imaginary component of real-time is $d\tau = idt$, c is the speed of light in Fig. 2. According to the special theory of relativity (principles of Einstein's), when two reference frames move relative to each other near the speed of light, they will measure different values for the length and the duration of events. This is due to the effects of time dilation $dt = \gamma dt'$ and length contraction, $dx' = \gamma dx$ (we know that the speed of light is constant in all IMFs and also all laws of physics are the same in all IMFs). Now we describe the dynamic motion of a particle. For all

particles that move with a velocity v comparable to c, their mass and energy increase according to the relation $E = m$ in the natural unit ($\hbar = c = 1$) ($E = mc^2$ in SI units). We now try to study and describe the behavior of a relativistic system, according to Section 3, using the behavior of a system as a behavior of a spring and dashpot in the Kelvin–Voigt model. The Kelvin–Voigt model is presented under these principles of relativistic physics: a) in all inertial frames of reference, the laws of physics are the same; b) absolute motion cannot appear in any law of physics; c) in all inertial frames of reference all experiments run the same, d) the absolute motion of the observer cannot be defined by any experiment; e) c is constant value in all inertial frames of reference; f) Nothing can travel faster than c, which is the upper limit for the speeds of objects with a positive relaxation mass (rest mass). Hence, in the relativistic limit of Einstein's theory, when a relativistic external force $F(t)$ along a given direction, x is applied to a system with the rest mass m_0, the system moves directly in a straight line along that direction x (i.e., in the direction of the relativistic force $F(t)$), and its moving mass m will change with the velocity v by the relation

$$m(t) = \gamma m_0 \tag{30}$$

and the relativistic force that increases velocity and mass reads [23,24]

$$F(t) = \frac{dp}{dt} = m(t)\frac{dv(t)}{dt} + v(t)\frac{dm(t)}{dt} \tag{31}$$

Origination based on mass–energy equivalence $E = m$, the mass of the system increases as the velocity increases. However, there is a specific condition on the increases in mass and velocity: "the speed of an object cannot reach c". Only massless objects, including photons, which make up light, can travel at c. Therefore, objects moving under a relativistic velocity, with a non-zero rest mass, have a specific limit on their relativistic velocity v. Based on the Kelvin–Voigt model, which combines a spring-dashpot element, we can explain the motion of this relativistic object under an external force $F(t)$. These two relativistic principles help us to study this problem: *i)* an increase in mass with increasing velocity, *ii)* a restriction or limitation on the relativistic velocity. Hence, one can consider that applying the external force requires a combination of two different types of forces,

which satisfy the conditions of increasing the mass with increasing velocity, force $F_m(t)$ and restricting the velocity in the limit condition of $F_v(t)$, as follows:

$$F(t) = F_m(t) + F_v(t) \tag{32}$$

First, we consider a relativistic object to be an elementary particle with radius r and volume V at rest, and that, during relativistic motion, its mass and volume reach dm and dV, within $v(t)$ and $\xi_R(t) = \frac{dv(t)}{dt}$. In the above conditions, the spring-dashpot model's unique quantities used to describe the external force, based on the spring and dashpot reactions, are $x(t)$ and $\dot{x}(t)$, i.e., the force $F_m(t)$ increases the displacement of the spring, and the force $F_v(t)$ decreases the elongation via the dashpot 's characteristics. As we considered above, in the relativistic motion of an object, that unique quantity can be the relativistic velocity of the object. Using equation (20), one can express modified equation (31) in the context of the Kelvin–Voigt model, in the following manner [28]

$$F(t) = F_m(t) + F_v(t) = \mu v(t) + \eta \dot{v}(t) \tag{33}$$

and then the changes of forces with respect to dt read

$$\dot{F}(t) = \mu \dot{v}(t) + \eta \ddot{v}(t) \tag{34}$$

As we showed in Section 3, by exerting $F_m(t)$, the mass acquires velocity; additionally, $F_v(t)$ acts as a damping force, preserving the velocity of the moving mass within the limiting condition of the relativistic velocity c, and leads to a reduction in the velocity of the moving mass. Equation (33) describes the theoretical formulation of the inherent memory mechanism of a relativistic object. As considered in the preceding paragraph, the applied external force as a function of time $F(t)$ gives the velocity $v(t)$and the oscillation $\dot{v}(t)$, in direction of applied force, to the object (or mass) through a direct proportionality relation that is itself a function of time. In relativistic motion, based on the velocity limitation of an object and using the Heaviside step function *H(x)* and its derivative, the Dirac delta function $\delta(x)$, during applying a sudden force $F(t) = aH(x)$, we determine constant parameter of the first part of the external force $F_m(t) = \mu v(t)$ in the natural unit ($\hbar = c = 1$), which is equal to

$\mu = a$ ($c\mu = a$ in the SI unit), and, which is presented as the linear spring constant in the Kelvin–Voigt model. Hence, the relativistic characteristic function $\beta_R(t)$ using equations (21-28) and the relation $\dot{v}(t) = \xi_R(t) = \beta_R(t) a\dot{H}(t)$ reads:

$$\beta_R(t) = c\left(1 - exp(-\frac{kt}{\eta})\right) \tag{35}$$

Comparing equations (31) and (33), one can define

$$F(t) = \frac{dm}{dt}v(t) + m(t)\,\dot{v}(t) = \mu v(t) + \eta\dot{v}(t) \tag{36}$$

Then, using $\int_0^{t_\infty} \delta(t)dt = 1$, and then we can define

$$\frac{dv(t)}{dt} = \beta_R(t)\frac{dH(t)}{dt} \Rightarrow v(t) = \left(1 - exp(-\frac{kt}{\eta})\right) \tag{37}$$

Therefore, one can define the mass of a relativistic object during motion by solving the resulting equation (36) using (35) as follows

$$F(t) = m(t)\,\frac{dv(t)}{dt} + v(t)\frac{dm}{dt} = av(t) + \eta\frac{dv(t)}{dt} \tag{38}$$

Using equation (37), under the condition of applying a sudden force $F(t) = aH(t)$, and we can define the damping coefficient of the dashpot η and the rest mass values, based on the Kelvin–Voigt model in the asymptotic limits of time t_0 and t_∞. Therefore, integrating equation (38) under the conditions $t = t_0$ and $t = t_\infty$ and defining independent equations:

$$1)\ \lim_{t\to 0}\int_{m_0}^{m} dm = \lim_{t\to 0}\int_0^{t_\infty}\frac{dt}{v(t)} \Rightarrow m_0 = \lim_{t\to 0}\frac{t}{\left(1-\exp(-\frac{kt}{\eta})\right)} = \frac{\eta}{a} \tag{39}$$

$$2)\ \lim_{t\to t_\infty}\int_{m_0}^{m} dm = \lim_{\substack{t\to t_\infty \\ v(t)\to const=1}}\int_0^{t_\infty}\frac{dt}{v(t)} \Rightarrow m(t_\infty) = \frac{t_\infty}{\left(1-exp(-\frac{t_\infty}{m_0})\right)} \tag{40}$$

Hence, originating from the memory model (the Kelvin–Voigt model [28-31,32]) and the relativistic behaviour of a moving object, the mass and velocity of an object at the time t during motion within the applied external force become as follows

$$m(t) = \frac{t}{\left(1-exp\left(-\frac{t}{m_0}\right)\right)}, \qquad v(t) = \left(1 - exp(-\frac{t}{m_0})\right) \tag{41}$$

From the above relation of $v(t)$, the time defines $t = -m_0 ln\big(1 - v(t)\big)$, and using the Taylor's expansion of $ln\big(1 - v(t)\big) \cong -v(t) - \frac{v(t)^2}{2} - \frac{v(t)^3}{3} - \cdots$, the time t determines

$$t \cong m_0 v(t) \left(1 + \frac{v(t)}{2} + \frac{v(t)^2}{3}\right) \tag{42}$$

5. Localized State of Two Particles in the Kelvin–Voigt Memory Model

In Section 2, we presented the localized state system of two relativistic particles with the rest masses, $m_0 = m_1 = m_2$, within the strong Coulomb interaction $V(r) = -\frac{\alpha_s}{r}$. Now, we explain the creation of a localized state under the Kelvin–Voigt memory model using a combination of two parallel springs and a dashpot as shown in Fig. 3. An external force can be considered as the gradient of the potential interactions, which gives rise to the localized state system of two particles [15,27,28]. In the schematic form of the memory model, these two particles can be represented as two springs with spring constant k, and a dashpot with a damping constant η [28].

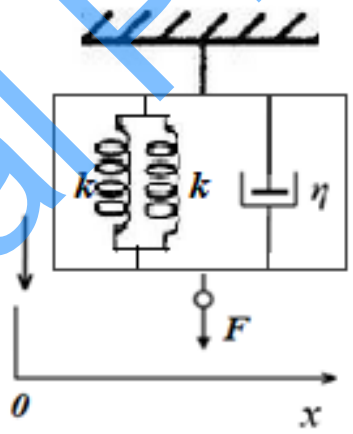

Fig. 3. The Kelvin–Voigt model of two parallel springs and a dashpot as a pionic atom.

Now, we define the velocity of the moving particles based on the relativistic effective quark mass μ_i and the rest mass of the quark m_i . Hence, originating from equation (20) in the relativistic QFT context and using the relativistic mass relation $\mu_i = \gamma m_i$, the relativistic velocity of a particle under the action of the external field in the natural unit ($\hbar = c = 1$), reads

$$v_i = \sqrt{1 - \left(\frac{m_i}{\mu_i}\right)^2} \tag{43}$$

and the relativistic relation $a_r = \frac{\mu_i}{m_i}$ based on Einstein's mass formula $\mu_i = \gamma m_i$ reads

$$a_r = 1 + \frac{1}{2}v_i^2 + \frac{3}{8}v_i^4 + \frac{5}{16}v_i^6 + \cdots \tag{44}$$

Then, the relation between the relativistic mass and the rest mass of an object $a_m = \frac{\mu_i}{m_i}$, based on the memory model and equations (41, 42), becomes

$$a_m = 1 + \frac{1}{2}v_i + \frac{1}{3}v_i^2 + \frac{1}{4}v_i^3 + \cdots \tag{45}$$

As shown above, from the equations in the preceding sections, for the pion π^+ and π^- with the rest masses $m_1 = m_2 = 136.80MeV$, in the Coulomb field within the constant of interaction $\alpha_s = 0.1 - 0.5$ the relativistic effective pion mass μ_π based on the relativistic QFT, is defined and calculated. The parameters a_r and a_m the relativistic QFT and the memory model of a coupled springs-dashpot system are calculated. The results are calculated for the ground state and the first excited state. The data are shown in Tables 1 and 2. We can compare the theoretical results obtained using relativistic quantum field theorya_r and the memory model a_m.

Table 1. The effective relativistic mass μ_π of the pions $\pi^\pm$, the ratio of $a = \mu_\pi/m_\pi$, and the velocity v_π of a pion in the ground state $\ell = 0$, with the coupling constant α_s in the natural unit.

α_s	$\mu_\pi(MeV)$	v_π	a_r	a_m	$\Delta a = a_m - a_r$
0.1	136.97	0.05	1.0013	1.0259	0.0246
0.2	137.49	0.10	1.0050	1.0536	0.0485
0.3	138.37	0.15	1.0114	1.0833	0.0719
0.4	139.62	0.20	1.0206	1.1153	0.0947
0.5	141.29	0.25	1.0328	1.1497	0.1169

Table 2. The effective relativistic mass μ_π of the pions $\pi^\pm$, the ratio of $a = \mu_\pi/m_\pi$, and the velocity v_π of a pion in the first excited state $\ell = 1$, with the coupling constant α_s in the natural unit.

α_s	$\mu_\pi(MeV)$	v_π	a_r	a_m	$\Delta a = a_m - a_r$
0.1	136.84	0.02	1.0003	1.0127	0.0124
0.2	136.97	0.05	1.0013	1.0259	0.0246
0.3	137.19	0.07	1.0028	1.0395	0.0367
0.4	137.49	0.10	1.0050	1.0536	0.0485
0.5	137.88	0.13	1.0079	1.0682	0.0603

The analytical results in the above tables describe the relativistic correction to the mass of a pionic atom based on the QFT and the memory model. The parameter $\Delta a = a_m - a_r$ shows the relativistic corrections based on the memory model. The memory model defines the relativistic corrections more clearly and with more precision. To determine this correction, we used the velocity value corresponding to the effective relativistic mass μ_π of the pion $\pi^\pm$ in the pionium localized states, and used this value as the primary input required for the calculations in the memory model. In **Fig. 4,** we present the parameters a_r and a_m as a function of velocity $v_\pi = 0.05, 0.1, 0.15, 0.2, 0.25,$ for the ground states under the different coupling constants of interaction, $\alpha_s = 0.1, 0.2, 0.3, 0.4, 0.5$ in the natural unit ($\hbar = c = 1$), respectively.

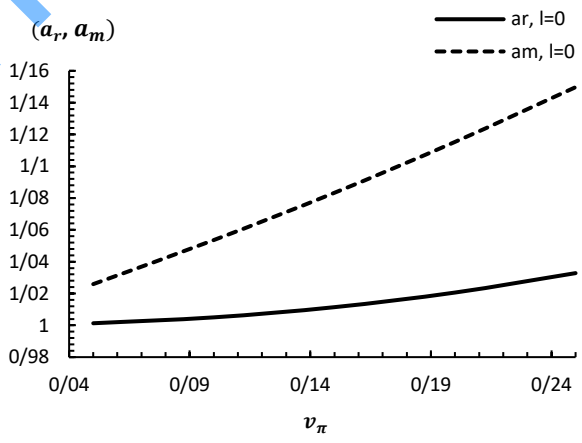

Fig. 4. Parameters a_r and a_m as a function of velocity $v_\pi = 0.05, 0.1, 0.15, 0.2, 0.25$, the ground states $\ell = 0$, with $\alpha_s = 0.1, 0.2, 0.3, 0.4, 0.5$, in the natural unit ($\hbar = c = 1$).

In Fig. 4, the curves show that the relativistic effects in the memory model have a stronger effect on the effective relativistic mass μ_π of the pions $\pi^{\pm}$ than QFT at the same velocity v_π; Δa increases with increasing velocity and the constant of interaction. The parameter a_m changes with a greater slope compared to a_r. The results have shown that a_m, can be labelled as a new, individual parameter at ultra-relativistic velocities of moving objects. We may consider that the results of this research, using the memory model in particle physics and high-energy physics, are expected to yield a new theoretical description of the hadronic localized states. This parameter a_m, can be important, especially when spin-orbit interactions are included, and these theoretical data are useful in guiding new research directions aimed at defining the properties of hadronic systems. Based on the above tables, we plot Fig. 5, showing the dependence of a_r and a_m as the relativistic correction to the pionic atom properties, for the ground state and the first excited state mass, as a function of the coupling constant.

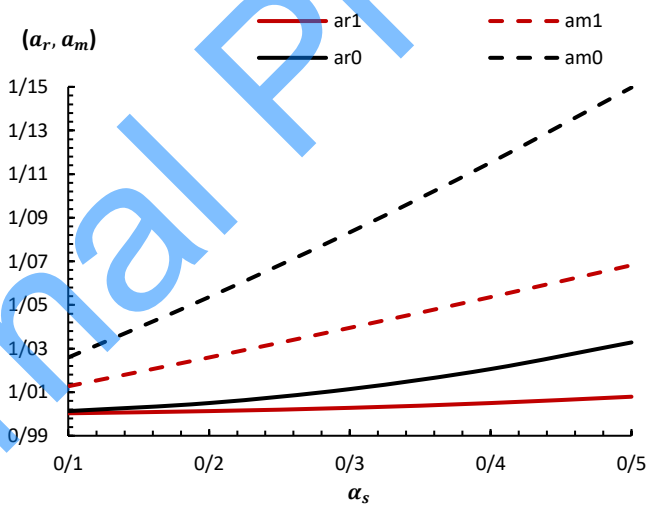

Fig. 5. Parameters a_r and a_m as a function of the coupling constant α_s, for $\ell = 0, 1$.

The relativistic correction to the velocity in the memory model based on the increasing mass of the moving object for the ground and excited states is calculated and presented in Fig 6. The parameter $\Delta v = v_m - v_r$, the difference between the value of the relativistic velocity based on the memory v_m model and the QFT v_r, is presented in Table 3.

Table 3. The parameter $\Delta v = v_m - v_r$ of the pion $\pi^{\pm}$, in the ground state $\ell = 0$ and the excited states $\ell = 1,2,3$.

	α_s		
	0.5	**0.3**	**0.1**
ℓ	$\Delta v = v_m - v_r$	$\Delta v = v_m - v_r$	$\Delta v = v_m - v_r$
0	0.0058423	0.0014052	0.0000586
1	0.0008367	0.0001917	0.0000076
2	0.0002604	0.0000586	0.0000023
3	0.0001127	0.0000251	0.0000010

These velocities v_r and v_m, which are plotted in **Fig. 6**, decrease as the orbital quantum number ℓ increases. This trend is shaped by the Coulomb external field; in the strong field and quark-gluon potential of interaction at high energies, this relationship may change.

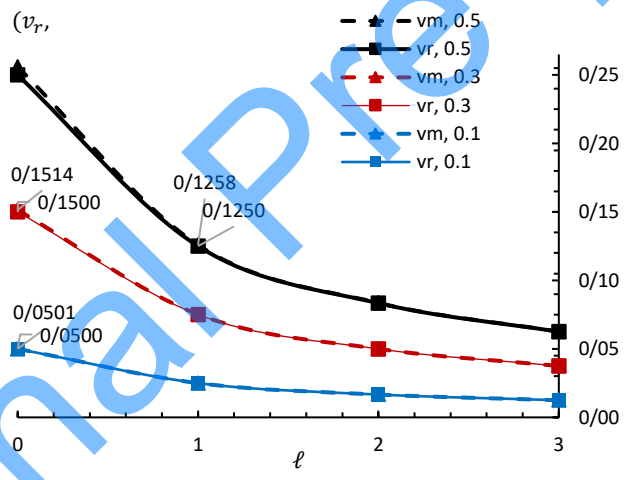

Fig. 6. Parameters v_r and v_m as a function of the orbital quantum number $\ell = 0$, 1,2,3 with the different coupling constant of interaction $\alpha_s = 0.1$, 0.3, 0.5.

6. Discussion

Pionium is a localized state of $\pi^{\pm}$ mesons. Theoretical calculations of the mass spectrum of the pionic atom and the effective relativistic mass of the $\pi^{\pm}$ based on modified equations in QFT and relativistic theory can determine various properties and characteristics of this exotic hadronic system. Here is some theoretical information that motivated us to study the pionic atom properties under the

relativistic conditions and QFT methods:

- Mass spectrum of a pionic atom: Theoretical calculations involve determining the masses of the $\pi^{\pm}$ based on the relativistic effects and corrections.
- Binding energy and energy eigenvalues: These quantities reflect the strength of the interaction between the constituents $\pi^{\pm}$ and the relativistic motion of the pion $\pi^{\pm}$ within the localized state.
- Decay width and the finite lifetime of a pionic atom: These properties must be determined under the strong interactions and relativistic limitations of mass and velocity.
- The wavefunction of a pionic atom: Mathematical calculations describe the spatial distribution of the constituents $\pi^{\pm}$ within the pionic atom. Clarifying and providing valuable insights into the properties and behavior of the pionic atom helps us to understand the underlying physics of these exotic hadronic localized states.

Hence, taking into account the dialogue on the relativistic QFT and the memory model of the spring-dashpot system in structural mechanics, the hadronic localized state properties of the pionic atom at the relativistic limit under the strong Coulombic interaction have been studied. Based on the memory model principles, we considered the relativistic object as a combination of a spring and a dashpot, which govern the relativistic behavior and the limitation conditions. This consideration led us to present the relativistic effect of external forces, which can act as loading effects on the deformation behavior of the springs and dashpots. This explanation describes how the movement of an object is mathematically connected to the intrinsic history of the loading force (or external force) on the spring and dashpot. Using the relativistic theory and the oscillating properties of the quantum harmonic oscillator (particle) in quantum field theory, we calculated the effective relativistic mass of the pion $\pi^{\pm}$ as the relativistic mass of a moving particle within the localized states of pionium. This mass correction at the relativistic velocity helped us establish a connection between QFT and the memory model. The results reveal that the relativistic corrections to the total mass, the energy eigenvalues, and the effective relativistic mass differ from those predicted by the standard relativistic

Einsteinian formalism. Although the differences $\Delta a = a_m - a_r$ a and$\Delta v = v_m - v_r$ are small, they indicate that the memory model produces relativistic corrections that systematically differ from those obtained within the standard Einsteinian framework. A quantitative comparison with experimental data will be the subject of future investigations.

Fig. 4 and Fig. 5 depict the curves of a_r and a_m, showing that relativistic corrections based on the memory model increase faster compared to the usual Einsteinian model as the velocity and the interaction constant increase. In Fig. 6, the curves of v_r and v_m demonstrate that, with an increase in the quantum number ℓ, v_m decreases at a slower rate than v_r. These behaviors can be explained based on the characteristics of the spring-dashpot model of the moving object. Specifically, as the quantum number ℓ, increases, the spring-dashpot memory model tends to retain the system's previous state more strongly, resulting in a slower reduction in velocity v_m compared to velocity v_r.

7. Conclusion

The study presents a physical farmwork to calculate the properties of the localized state of two particles. The properties are derived from quantum field theory, relativistic effects, and a memory model. The model describes the relativistic effective mass of the constituent particles relative to their rest mass, in the high-energy limit, under the combination of the quantum harmonic oscillator and the Kelvin-Voigt memory model. The theory uses Boltzmann's constitutive equations for a pionic atom under the strong interaction, which introduces a kind of elastic memory into the dynamics of the system. The application to pionic atoms is consistent with experimental data. This type of theoretical research is of interest in modern particle physics because it may lead to new effective theories of hadronic physics.

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manuscript. All intellectual content and computational results on the mass spectrum of the localized state, however, remain the sole responsibility of the author.

Data availability

All data reported in this article were obtained from the analytical equations discussed within the article and are available upon request by contacting the corresponding author of the study.

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