

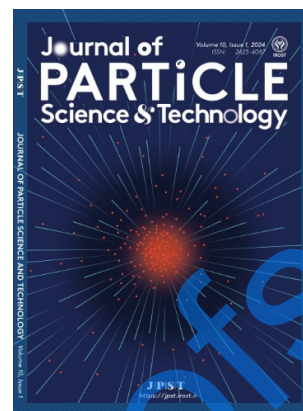
## Journal Pre-proofs

Effects of hybrid confinement potential on predictable hadronic highly resonance states

Arezu Jahanshir, Ekwevugb Omugbe

DOI: <https://doi.org/10.22104/jpst.2025.7612.1277>

Manuscript number: JPST-2505-1277



To appear in: *Journal of Particle Science and Technology (JPST)*

Received Date: 11 May 2025

Received Date in revised form: 11 June 2025

Accepted Date: 25 June 2025

**Please cite this article as:** Jahanshir A., Omugbe E., Effects of hybrid confinement potential on predictable hadronic highly resonance states, *Journal of Particle Science and Technology* (2024), doi: <https://doi.org/10.22104/jpst.2025.7612.1277>

File of an article that has undergone enhancements after acceptance, such as the addition of a cover page and metadata, and formatting for readability, but it is not yet the definitive version of record. This version will undergo additional copyediting, typesetting and review before it is published in its final form, but we are providing this version to give early visibility of the article. Please note that, during the production process, errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

© 2024 The Author(s). Published by [IROST](#).

# Effects of Hybrid Confinement Potential on Predictable Hadronic Highly Resonance States

Arezu Jahanshir<sup>1,\*</sup>, Ekwevugb Omugbe<sup>2</sup>

<sup>1</sup>Department of Physics and Engineering Sciences, Buein Zahra Technical University, Iran

<sup>2</sup>Department of Physics, University of Agriculture and Environmental Sciences, Nigeria

## Abstract

Unlike light quarkonium bound states, the relativistic effects in highly massive quarkonium states such as Bottomia, in the large radial excitation states, cause Bottomia to have a strong reaction and sensitivity to relativistic corrections. Highly excited states have a larger separation distance between the constituent quark-antiquark pair in the bound state. Therefore, they will have a relatively high velocity, which makes the relativistic kinetic energy and relativistic mass of quarks non-negligible. Notably, in this study, one of the important behaviors of relativistic effects of predicted highly excited hadronic bound states of Bottomia within the hybrid confinement potential is obtained. The high energy refinement and relativistic effect modification of mass and kinematic energy in the formalism of quantum oscillator principles and quantum field theory under the auxiliary variational method are defined. Relativistic corrections and effects on the shape of potential due to relativistic mass are plotted and compared to non-relativistic plots that are calculated. Values of the mass spectra of Bottomia used in the potential plot are consistent with predictions from other theoretical approaches and explain results behaviors in the used method.

**Keywords:** Highly excited states; Hybrid potential; Relativistic correction; Variational method.

## 1. Introduction

The different spin arrangement properties of angular momentum states in predicted highly excited Bottomia bound states ( $b - b^*$ ), among other highly resonance states, are important and significant [1,2]. These highly massive resonance states have been distinguished based on the theoretical behaviors of the standard quark model, allowing us to explore different and various phenomena of bound states. The hybrid confinement potential (HCP) model in quantum chromodynamics (QCD), plays an important role in reviewing and analyzing the characteristics of

strongly excited resonance states of Bottomia. These types of potentials help us with understanding the perturbative and nonperturbative Hamiltonian of interactions, particle dynamics, decay rate, bound state properties, and also quark-gluon dynamics in high energy physics and ultra hot medium. In the last experimental research from the LHC, Super KEKB accelerator (Belle II), ALICE Experiment, LHCb Experiment, CMS and ATLAS Experiments, the highly resonant states of Bottomia, such as  $7s: Y(10860)$  and  $10s: Y(11020)$  can be better described and analyzed using HCP model [3,4]. The calculated spectra of Bottomia, based on Coulombic and confinement characteristics of potential models, closely align with experimental results obtained from strong interaction. The bound states' mass spectroscopy, decay channels, transition rates, hybrid states, relativistic corrections on mass and nonperturbative potential, wavefunction analysis, and decay mechanism under extreme ultra hot temperature conditions of the environment in high energy physics can be investigated better within HCP models. Hence, predicting new resonance states of Bottomia and new exotic hadronic bound states is enabled by determining mass spectra under a specific potential model. This intertwined space of experimental and theoretical knowledge regarding additional excitations and more complex hadronic states is filled and completed by studying the properties of Bottomia bound states. Bottomia based on relativistic and nonperturbative properties is the best candidate for investigating hybrid states because most of the expected behaviors of its resonance states have certain anomalies [5]. These anomalies can describe potential impact factors and transition rates. As we know, recent research and studies have focused on the relativistic behaviors of Bottomia in highly excited states, which leads to improved theoretical calculation of mass spectra using relativistic mass. Hence, in this article, we define relativistic corrections on mass and potential interaction in Bottomia bound states. The paper is organized as follows: In section 2, we determine the mass and constituent mass of Bottomia bound states, based on canonical variable transformations in symplectic space under the oscillator representation method, where the wavefunction is presented as a Gaussian basis function. In section 3, the relativistic radial Schrödinger equation (RRSE) using relativistic energy

within the HCP is solved. In section 4, we plot effective potential shape under a variation of effective potential parameters  $\mu_b, \xi$  with the nonrelativistic and relativistic RSE, defining relativistic corrections on mass spectra and eigenenergies of excited states. In the last section, the practical application of the results of theoretical calculations of the mass spectrum and interaction potential behavior is demonstrated. We use computational coding and graph visualization with *MATLAB R2021a* software, and for additional mass spectra calculations, we use Excel 2022 software.

## 2. Relativistic form of the Schrödinger equation

We have defined the bound state properties using quantum field theory and the Feynman path integral method within the formalism of a gauge boson interaction with  $b - b^*$  pairs, based on the theoretical definition of the vacuum polarization corrections that connect different loop diagrams in Feynman path integrals [6-8]. The nonrelativistic and relativistic radial Schrödinger equation reads:

$$H\Psi(r) = E(\mu)\Psi(r) \quad (1)$$

$$H_{rel}\Psi(r) = E(\mu)\Psi(r) \quad (2)$$

Therefore, for  $b - b^*$  pairs bound state has the form non-RRSE has the form:

$$\left( \frac{\hat{p}_r^2}{2m_b^2} + \frac{\hat{p}_r^2}{2m_{b^*}^2} + V(r) \right) \Psi(r) = E(m_b, m_{b^*}) \Psi(r) \quad (3)$$

It can easily be solved by many different methods. By modifying (1), one has RRSE model as the following form:

$$H_{rel}\Psi(r) = E(\mu)\Psi(r)$$

and it is defined by changing classical kinetic energy to the total relativistic energy:

$$\frac{1}{2}m_0v^2 \rightarrow \sqrt{m_0^2 + p_r^2}$$

where  $m_0$  is the rest mass and  $p_r$  is the relative radial momentum of a quark-antiquark pair in a bound state. On the other hand, as we know that the inequality of the arithmetic and geometric inequality

equation in mathematics holds for two non-negative numbers  $x$  and  $y$  with equality if and only if  $x = y$ , as follows:

$$\frac{x + y}{2} \geq \sqrt{xy}$$

Choosing  $x = \mu$  and  $y = \frac{m_0^2 + \hat{p}_r^2}{\mu}$ , then:

$$f(\mu^*) \geq \sqrt{\mu^* \cdot \frac{m_0^2 + \hat{p}_r^2}{\mu^*}} = \sqrt{m_0^2 + \hat{p}_r^2}$$

One considers that the arithmetic mean of a set of non-negative real numbers is greater than or equal to the geometric mean of the same set. The minimal (optimal) choice of parameter  $\mu^*$  is defined when  $\mu^* = \frac{m_0^2 + \hat{p}_r^2}{\mu^*}$ , therefore, we have

$$\mu^{*2} = m_0^2 + \hat{p}_r^2 \rightarrow \mu^* = \sqrt{m_0^2 + \hat{p}_r^2}$$

and one can define the minimum value of the function  $f(\mu)$  within the minimal choice of  $\mu^* = \mu$ , which is equal to:

$$\min_{\mu > 0} f(\mu) = \sqrt{m_0^2 + \hat{p}_r^2}$$

Inserting the minimal value  $\mu$  back in the first relationship,

$$\sqrt{m_0^2 + \hat{p}_r^2} = \frac{1}{2} \left( \sqrt{m_0^2 + \hat{p}_r^2} + \frac{m_0^2 + \hat{p}_r^2}{\sqrt{m_0^2 + \hat{p}_r^2}} \right)$$

Using this equation, we define the effective mass approximation in high energy physics in the form of:

$$\sqrt{m_0^2 + \hat{p}_r^2} = \min_{\mu > 0} \frac{1}{2} \left( \mu + \frac{m_0^2 + \hat{p}_r^2}{\mu} \right)$$

where the minimal value choice of this parameter  $\mu$  is given for the mass of the particle. Below, we consider this optimal value as a relativistic mass of a particle in the bound states or the constituent mass of particles in the context of high energy physics. Now, we use a mass approximation in (2) for

the bound states of the pair  $b - b^*$  within the potential of interaction  $V(r)$  with the rest masses  $m_b = m_{b^*} = m_0$  and the relative momentum  $|\hat{p}_{rb}^2| = |\hat{p}_{rb^*}^2| = |\hat{p}_r^2|$

$$\left( 2 \min_{\mu_b} \frac{1}{2} \left( \mu_b + \frac{m_0^2 + \hat{p}_r^2}{\mu_b} \right) + V(r) \right) \Psi(r) = E(\mu_b) \Psi(r) \quad (4)$$

parameter  $\mu_b = \mu_{b^*}$  is the relativistic mass correction of the rest mass of the pair  $b - b^*$  in the bound states (constituent mass of particles). Now, for describing two-particle interactions and neglecting the spin interactions, we use the principles of quantum mechanics (QM) and quantum field theory (QFT). As we know, the probability amplitude for a particle's (the pair  $b - b^*$ ) transformation from  $x_b, x_{b^*}$  to  $y_b, y_{b^*}$  under the external quantum gauge field  $A_\alpha$  by the propagator function is described [7,8]. The initial and final states of  $b - b^*$  bound states are connected by this function and determine how the bottom and antibottom quarks travel over spacetime in the Minkowski coordinate system. In QM and QFT, the Green's function is presented as a propagator in the form [9]

$$G(x_b, x_{b^*}; y_b, y_{b^*} | A_\alpha) = \langle y_b, y_{b^*} | e^{-iHt} | x_b, x_{b^*} \rangle$$

and, the interaction of the pair  $b - b^*$  is described by  $\Pi(x_b, x_{b^*}; y_b, y_{b^*})$  polarization loop operator

$$\Pi(x_b, x_{b^*}; y_b, y_{b^*}) = \langle G_b(x_b, x_{b^*}; y_b, y_{b^*} | A_\alpha) G_{b^*}(x_b, x_{b^*}; y_b, y_{b^*} | A_\alpha) \rangle_{A_\alpha}$$

and the field propagator by Green's function

$$\delta^{(4)}(x_b, x_{b^*}) \delta^{(4)}(y_b, y_{b^*}) = \left[ (i\gamma_b^\alpha \partial_{\alpha,b} + igA_\alpha) - m_b^2 \right] G(x_b, x_{b^*}; y_b, y_{b^*} | A_\alpha) + \left[ (i\gamma_{b^*}^\alpha \partial_{\alpha,b^*} + igA_\alpha) - m_{b^*}^2 \right] G(x_b, x_{b^*}; y_b, y_{b^*} | A_\alpha) + V(x_b, x_{b^*}; y_b, y_{b^*} | x_b, x_{b^*}; y_b, y_{b^*}) \quad (5)$$

Eq. (3) describes an effective interaction of two particles for creating the bound states in an external field within a potential interaction in QFT [9,10]. We have to average and integrate over the external field to determine properties and all possible configurations of interaction using Eq. (3). This equation explains the external field reaction to a disturbance of the bounded particles at  $y_b, y_{b^*}$ . Based on the context of quantum chromodynamics (QCD) in high energy limits, we consider the long-range

forces  $((x_b, x_{b^*}) - (y_b, y_{b^*}))$  that lead to creating a Bottomia bound state with the mass  $M$ , and rewrite the behavior of the polarization loop operator in the form [11, 12]

$$\Pi(x_b, x_{b^*}; y_b, y_{b^*}) = \int \frac{d^4}{(2\pi)^4} \frac{1}{p^2 - M^2 + i\varepsilon} e^{ip(x_b, x_{b^*}; y_b, y_{b^*})} \quad (6)$$

As the pair  $b - b^*$  loop function  $\Pi(x_b, x_{b^*}; y_b, y_{b^*})$  decays exponentially this gives us a functional equation to calculate the mass spectra

$$\Pi(x_b, x_{b^*}; y_b, y_{b^*}) = C \cdot e^{-M|(x_b, x_{b^*}) - (y_b, y_{b^*})|}$$

where  $C$  is a normalization constant. Its value can be determined from the properties of the wavefunction of the bound state, boundary conditions of the bound state, and the interaction kernel. The specific conditions  $M < \infty$  and  $M = m_b + m_{b^*}$  are the necessary conditions to create a bound state. Now the polarization loop operator using QM principles based on the Feynman path-integral formulation by modifying equations for the Green's function [6-8] in the external gauge field  $A_\alpha$  for the two-point correlation functions of fields with the rest masses  $m_b, m_{b^*}$

$$\Pi(x|A) = \int_0^\infty \int_0^\infty \frac{d\mu_b d\mu_{b^*}}{(8x\pi^2)^2} I(\mu_b, \mu_{b^*}) e^{-\frac{|x|}{2} \left[ \left( \frac{m_b^2}{\mu_b} + \mu_b \right) + \left( \frac{m_{b^*}^2}{\mu_{b^*}} + \mu_{b^*} \right) \right]}$$

where  $x = (x_b, x_{b^*}) - (y_b, y_{b^*})$ , parameter  $\mu_b, \mu_{b^*}$  are the constituent masses of particles in the bound states. These masses are presented as the relativistic mass of particles based on Eq. (2). After some mathematical corrections the mass spectrum of the bound state is defined

$$M = \lim_{|x| \rightarrow \infty} \frac{-\ln \Pi(x)}{|x|}$$

and after some mathematical adjustments and using the saddle point method from Eq. (2), the mass spectrum of the pair  $b - b^*$  bound state based on the rest mass  $m_b = m_{b^*}$ , the constituent mass  $\mu_b = \mu_{b^*}$  and  $E(\mu_b)$  energy of bound states reads [13,14]

$$M = \min_{\mu_b} \left( \frac{m_b^2}{\mu_b} + \mu_b + E(\mu_b) \right) \quad (7)$$

and minimizing this equation, we define the constituent mass equation of the quark as follows

$$\mu_b = \sqrt{m_b^2 - \mu_b^2 \frac{dE(\mu_b)}{d\mu_b}} \quad (8)$$

Hence, the interaction the pair  $b - b^*$  that contains potential and nonpotential terms is modified under the Feynman path integral (functional integral) in nonrelativistic theory reads

$$I(\mu_b) = C^* e^{-x E(\mu_b)}$$

where  $C^*$  is a constant and depends on the normalization and boundary conditions of the bound state.

### 3. Relativistic correction on the interaction potential

In this section, we present the relativistic mass correction and effects on the energy eigenvalue within HCP of 7s:  $\Upsilon(10860)$  and 10s:  $\Upsilon(11020)$ . Starting from Eq. (2) and solving RRSE under the Gaussian basis function method (GBM) for  $b - b^*$  bound states within the HCP of the form  $V(r) = -V_0 \frac{e^{\alpha r} - e^{-\alpha r}}{e^{\alpha r} + e^{-\alpha r}} + \sigma r$ , where  $V_0 > 0$ ,  $\alpha$ , and  $\sigma$  are constant with the specific values that are adjusted to match the experimental data of highly states of Bottomia [15]. HCP is important in high energy physics because of the complex interplay between two specific forces at short and long distances in QM, QFT, spectroscopy, and quark-antiquark interactions. Choosing HCP in hyperbolic  $\tanh(ar)$  or exponential forms with the confinement term, we can explain the balance between short and long interaction range [16]. The radial excitation and orbital excitation based on the physical structure of the quarkonium system can better describe theoretical prediction and experimental data under this type of potential. Using GBM or effective field models, HCP can refine spectral interpretation states [17]. This potential allows us to include a discussion on nonperturbative effects in strong coupling interactions in hadronic states and analyze the behaviors of  $b - b^*$  bound states under different kinds of interactions. GBM allows us to define and determine relativistic effects of a bound state problem in QCD and hadronic spectroscopy at high energy physics by setting the pair  $b - b^*$  as a quantum harmonic oscillator bound state. GBM helps us to represent canonical variables  $\hat{x} = \frac{(\hat{a}^+ + \hat{a})}{\sqrt{2m\omega}}$  and  $\hat{p} =$



$\frac{i\sqrt{2m\omega}(\hat{a}^+ - \hat{a})}{2}$  in the form of creation  $\hat{a}^+$  and annihilation  $\hat{a}$  operators that act on the ground state with the rest mass  $m$ , and  $\omega$  is the angular frequency of a particle in the quantum harmonic oscillator (QHO) model in QM, then we can describe the quantization of the Gaussian form of wavefunctions in a secondary space and extend it to QCD [18,19]. Hence, using the canonical variables presentation and potential interaction based on QFT forms by integral representations

$$\hat{\phi}(x) = \int dk \hat{a} e^{ik\hat{x}}$$

$$\hat{\phi}^+(x) = \int dk \hat{a}^+ e^{-ik\hat{x}}$$

$$V(x) = \int \left(\frac{dk}{2\pi}\right)^n \tilde{V}(k^2) e^{\left(-\frac{k^2}{4\omega}\right)} : e^{ik\frac{(\hat{a}^+ + \hat{a})}{\sqrt{2m\omega}}} :$$

$V(x)$  is the potential interactions under Wick's spacetime ordering operators principles, i.e., “: ■ :”.

Now, we consider the transformation of radial and momentum variables  $r, p_r$  of  $R^n$  space to the new axillary variable  $s^{2\xi}, p_s$  of  $R^\eta$ . This axillary space is the secondary space under transformation  $r = s^{2\xi}, p_r \rightarrow p_s$ , then the wavefunction can be transformed  $\Psi(r) \rightarrow q^{2\xi(n_\omega - 2n_r)} \Phi(s^2)$ , where the secondary space has a dimension

$$\eta = 4s(n_\omega - 2n_r) + 2\xi + 2 = 4\ell\xi + 2\xi + 2$$

and  $n_\omega = 0, 1, 2, \dots$  is the main quantum number in QHO method,  $n_r$  is the radial excitation number  $n_r$  and  $\xi$  is a variational parameter and allows us to achieve the physical features of the asymptotic behavior of the bound state. To define the mass spectrum of bound states in QFT we use QHO and QCD by explaining bound state characteristics under the field conditions such as  $\frac{d\varepsilon(\mu_b)}{d\xi} = 0$ , which is described in the next paragraph [9]. Hence, we have to define the parabolic form of canonical operators with the normal ordering method in the secondary space, and also consider that the total Hamiltonian of the pair  $b - b^*$  bound states interaction does not contain the core parabolic terms in the canonical variables. These conditions are a specific and main point of QHO method in the GBM

model and lead us to calculate the oscillator frequency  $\omega_{n_r}$  of the bound state. For the growing potential with the core parabolic and Wick-ordered correction terms, we determine the parabolic form of canonical operators in the Wick ordering method and then use it in the potential terms. Hence, in this paper, we consider only the core parabolic because the nonperturbative and spin interactions are not included in our calculation. Hence, the parabolic form of the position operator is defined with the Gemma function

$$s^{2\tau} = \omega_{n_r}^{-\tau} \Gamma\left(\frac{\eta}{2} + \tau\right) \Gamma^{-1}\left(\frac{\eta}{2}\right)$$

and momentum operator is defined as follows

$$p_s^{2\tau} = \omega_{n_r}^{\tau} \Gamma\left(\frac{\eta}{2} + \tau\right) \Gamma^{-1}\left(\frac{\eta}{2}\right)$$

where  $\tau = 1, 2, 3, \dots$ , the notation and canonical operators satisfy the commutator form  $[\hat{s}, \hat{p}_s] = i\hbar$ ,  $[\hat{a}, \hat{a}^+] = \eta$ . The radial Laplacian operator of RRSE in the secondary space  $R^\eta$

$$(H_{rel} - E(\mu_b))\Psi(r) = 0$$

under transformation  $r = s^{2\xi}$  reads

$$\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} \rightarrow \frac{d^2}{ds^2} + \frac{\eta-1}{s} \frac{d}{ds}$$

and then RRSE of  $b - b^*$  bound states within HCP interaction present as follows

$$\left\{ \frac{p_s^2}{2} + V(s^{2\xi}) - E(\mu_b) \right\} \Phi(q^2) = 0 \quad (9)$$

As we know, the spin Hamiltonian is highly significant. However, the primary focus of this article is on defining the relativistic effects of bound states using the GBM and QHO methods. Therefore, in the supplementary calculations, the spin Hamiltonian  $H_s = H_{SS} + H_{LT} + H_{TT}$  has not been incorporated, and its additional effects on the mass spectrum have been neglected. Now, using the parabolic form of canonical operators, Eq. (9), RRSE for the excited radial states in secondary space under the second quantization within  $V(r) = -V_0 \frac{e^{ar} - e^{-ar}}{e^{ar} + e^{-ar}} + \sigma r$  HCP is defined in the following way [9].

$$\varepsilon(\mu_b) = \left\{ \frac{\eta\omega_\ell}{4} + 2\mu_b\xi^2 s^{4\xi-2} \left( -V_0 \frac{e^{(\alpha s^{2\xi})} - e^{-(\alpha s^{2\xi})}}{e^{\alpha(\alpha s^{2\xi})} + e^{-\alpha(\alpha s^{2\xi})}} + \sigma s^{2\xi} \right) - 2\mu_b\xi^2 s^{4\xi-2} E(\mu_b) + n_\omega \omega_\ell \right\} \Phi(q^2) = 0 \quad (10)$$

The ground state  $n_\omega = n_r = 0$ , mass spectrum by expanding HCP under the Bernoulli series and the parabolic form of the position operator  $s^{2\tau}$  and using Eq. (7) reads [20,21].

$$M = \left( 4m_b^2 - 2\mu_b^2 \frac{dE(\mu_b)}{d\mu_b} \right)^{1/2} + \frac{\mu_b}{2} \frac{dE(\mu_b)}{d\mu_b} + E(\mu_b) \quad (11)$$

where the ground state energy eigenvalue reads:

$$E_0(\mu_b) = \frac{\omega_0^{2\xi} \Gamma^{-1}(\frac{\eta}{2} + 2\xi + 1) \Gamma(\frac{\eta}{2} + 1)}{8\mu_b \xi^2} + \frac{(\sigma - \alpha V_0) \Gamma^{-1}(\frac{\eta}{2} + 2\xi + 1) \Gamma(\frac{\eta}{2} + 3\xi + 1)}{\omega_0^\xi} + \frac{(\alpha^3 V_0) \Gamma^{-1}(\frac{\eta}{2} + 2\xi + 1) \Gamma(\frac{\eta}{2} + 5\xi + 1)}{3\omega_0^{3\xi}} \quad (12)$$

then minimizing:

$$\frac{d\varepsilon(\mu_b)}{d\xi} = 0 \text{ and } \frac{d\varepsilon(\mu_b)}{d\omega} = 0$$

the ground state oscillator frequency  $\omega_0$  and energy eigenvalue function is defined using the relation

$$\frac{dE(\mu_b)}{d\mu_b} = - \frac{\omega_0^{2\xi} \Gamma^{-1}(\frac{\eta}{2} + 2\xi + 1) \Gamma(\frac{\eta}{2} + 1)}{8\mu_b^2 \xi^2} \text{ in the above equations, then define the oscillator frequency } \omega_0 \text{ as}$$

follows:

$$\omega_0^{5\xi} - \frac{4(\sigma - \alpha V_0) \mu_b \xi^2 \Gamma(\frac{\eta}{2} + 3\xi + 1)}{\Gamma(\frac{\eta}{2} + 1)} \omega_0^{2\xi} - \frac{4(\alpha^3 V_0) \mu_b \xi^2 \Gamma(\frac{\eta}{2} + 5\xi + 1)}{\Gamma(\frac{\eta}{2} + 1)} = 0 \quad (13)$$

And the constituent mass as a relativistic effect on mass in the vacuum state  $\mu_b$  reads

$$\mu_b = \left( m_b^2 + \frac{\Gamma(\frac{\eta}{2} + 1) \omega_0^{2\xi}}{16\xi^2 \Gamma(\frac{\eta}{2} + 2\xi + 1)} \right)^{1/2} \quad (14)$$

Now, based on (10) and (12), we can define the mass spectrum, energy eigenvalue, and oscillator frequency in the highly radial resonance states  $n_r = \frac{n_\omega - \ell}{2}$  by equations [22]:

$$E_{n_r}(\mu_b) = \left( \frac{1}{8} + \frac{n_r}{\eta} \right) \frac{\Gamma(\frac{\eta}{2} + 1)}{\mu_b \xi^2 \Gamma(\frac{\eta}{2} + 2\xi + 1)} \omega_{n_r}^{2\xi} + \frac{(\sigma - \alpha V_0) \Gamma(\frac{\eta}{2} + 3\xi + 1)}{\Gamma(\frac{\eta}{2} + 2\xi + 1) \omega_{n_r}^\xi} + \frac{(\alpha^3 V_0) \Gamma(\frac{\eta}{2} + 5\xi + 1)}{3\Gamma(\frac{\eta}{2} + 2\xi + 1) \omega_{n_r}^{3\xi}} \quad (15)$$

and

$$\omega_{n_r}^{5\xi} - \frac{(\sigma - \alpha V_0)\mu_b \xi^2 \Gamma(\frac{\eta}{2} + 3\xi + 1)}{(\frac{1}{8} + \frac{n_r}{\eta})\Gamma(\frac{\eta}{2} + 1)} \omega_{n_r}^{2\xi} - \frac{(\alpha^3 V_0)\mu_b \xi^2 \Gamma(\frac{\eta}{2} + 5\xi + 1)}{4(\frac{1}{8} + \frac{n_r}{\eta})\Gamma(\frac{\eta}{2} + 1)} = 0 \quad (16)$$

The constituent mass  $\mu_b$  of the highly bound states reads

$$\mu_b = \left( m_b^2 + 2\left(\frac{1}{8} + \frac{n_r}{\eta}\right) \frac{\Gamma(\frac{\eta}{2} + 1)\omega_{n_r}^{2\xi}}{\xi^2 \Gamma(\frac{\eta}{2} + 2\xi + 1)} \right)^{1/2} \quad (17)$$

Now we calculate the mass spectra of Bottomia using the nonrelativistic (1) and relativistic (2) equations [11], and the results of Bottomia bound states in QHO and GBM model are presented using specific parameters [23] in Table 1.

$$m_b = m_{b^*} = 4.823 \text{ GeV}$$

$$V_0 = 1.1 \text{ GeV}, \quad \alpha = 0.1 \text{ GeV}$$

$$\sigma = 0.25 \text{ GeV}^2$$

We choose parameters to adjust the calculation of the mass spectrum based on theoretical data from other sources and experimental results. However, these parameters are defined only at the extreme point of bound state formation and are not adjusted in this calculation for long-range stability of bound states, and also, we consider  $a = \sigma - V_0\alpha > 0$ ,  $a \neq 0$ ,  $V_0 \neq 0$  which are extreme values solely for the creation of bound states, especially for highly resonant states. Hence, we can use experimental data and compare these values. Despite the predicted heavy mass of  $b - b^*$  in the highly radial excitation  $n_r = 7, 8, 9, 10$  states such 7s:  $\Upsilon(10860)$  and 10s:  $\Upsilon(11020)$ , the relativistic effects become more apparent, leading to significant internal motion and increasing the relativistic behavior of  $b - b^*$  bound state. We defined quantitative error analysis using Root Mean Square Error (RMSE), which measures the difference between the relativistic and nonrelativistic values based on our approach to calculating the masses of bound states and comparing them with experimental data. All values in our calculations are in  $\text{GeV}$ .

**Table 1.** Comparison of the mass spectrum in the lower and higher states of Bottomia

	$M_{nonrel}$	$M_{rel}$	$\mu_b$	$M_{Th}$	$M_{Exp}$	RMSE <i>nonrel</i>	RMSE <i>rel</i>
<b>1s: Y(94603)</b>	9.2692	9.4582	4.8390	9.463 [25-27]	9.4603 [11]	0.1913	0.0021
<b>2s: Y(10023)</b>	10.2354	10.3068	4.9048	10.023[23,28]	10.023 [24]	0.2125	0.2840
<b>7s: Y(10860)</b>	10.6709	10.8424	5.0039	10.887 [23]	10.885 [24]	0.2144	0.0421
<b>10s: Y(11020)</b>	10.8296	11.0522	5.0513	11.021 [23]	11.020 [24]	0.1904	0.0322

In this paper, we study this effect on the mass spectra and describe how this correction affects the shape of the potential interaction. In this article, we include in our calculation only the kinetic and the mass-dependent terms, and the effective potential related to the relativistic mass correction. The effective potential that is described in this research under QHO and GBM models based on the Schrödinger equation

$$\left( \frac{-1}{2\mu_b} \left( \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} \right) + \frac{\ell(\ell+1)}{2\mu_r r^2} - V_0 \frac{e^{\alpha r} - e^{-\alpha r}}{e^{\alpha r} + e^{-\alpha r}} + \sigma r \right) \Psi(r) = E(\mu_b) \Psi(r) \rightarrow$$

$$\left\{ \frac{p_s^2}{2} + 2\mu_b \xi^2 s^{4\xi-2} \left( -V_0 \frac{e^{(\alpha s^{2\xi})} - e^{-(\alpha s^{2\xi})}}{e^{(\alpha s^{2\xi})} + e^{-(\alpha s^{2\xi})}} + \sigma s^{2\xi} \right) \right\} \Phi(q^2) = E(\mu_b) \Phi(q^2)$$

determines as follows

$$V_{eff}(s) = 2\mu_b \xi^2 s^{4\xi-2} \left( -V_0 \frac{e^{(\alpha s^{2\xi})} - e^{-(\alpha s^{2\xi})}}{e^{(\alpha s^{2\xi})} + e^{-(\alpha s^{2\xi})}} + \sigma s^{2\xi} \right) \quad (18)$$

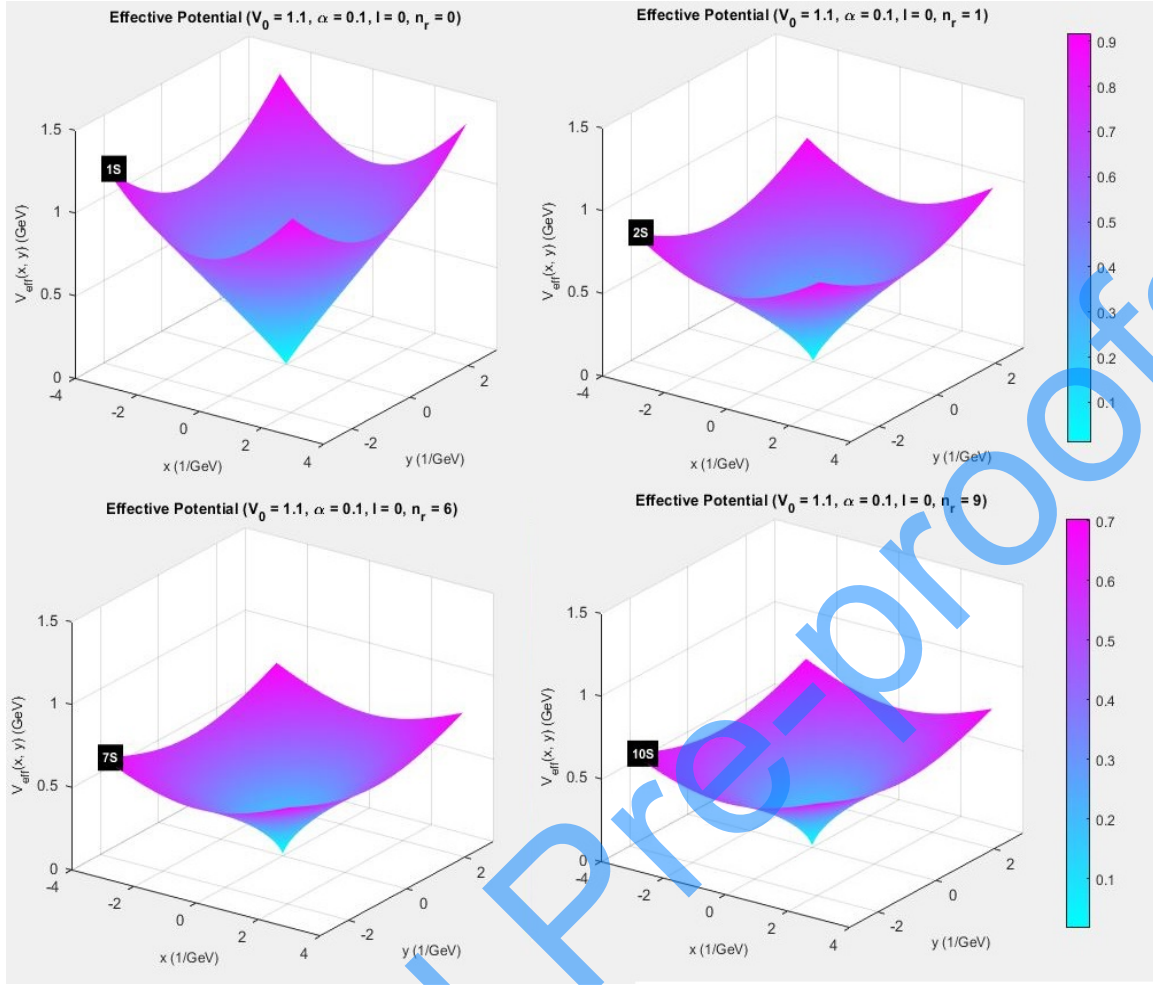
and presents a short-range attractive term and a long-range confinement term including the effect of  $\ell$  - the angular momentum quantum number, that appears in given parameters  $\mu_b$ ,  $\xi$  in the following form

$$\mu_b = \left( m_b^2 + 2 \left( \frac{1}{8} + \frac{n_r}{4\ell\xi + 2\xi + 2} \right) \frac{\Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 1) \omega_{n_r}^{2\xi}}{\xi^2 \Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 2\xi + 1)} \right)^{1/2} \quad (19)$$

$$\frac{d}{d\xi} \left[ \left( \frac{1}{8} + \frac{n_r}{4\ell\xi + 2\xi + 2} \right) \frac{\Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 1)}{\mu_b \xi^2 \Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 2\xi + 1)} \omega_{n_r}^{2\xi} + \frac{(\sigma - \alpha V_0) \Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 3\xi + 1)}{\Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 2\xi + 1) \omega_{n_r}^\xi} + \frac{(\alpha^3 V_0) \Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 5\xi + 1)}{3 \Gamma(\frac{4\ell\xi + 2\xi + 2}{2} + 2\xi + 1) \omega_{n_r}^{3\xi}} \right] = 0 \quad (20)$$

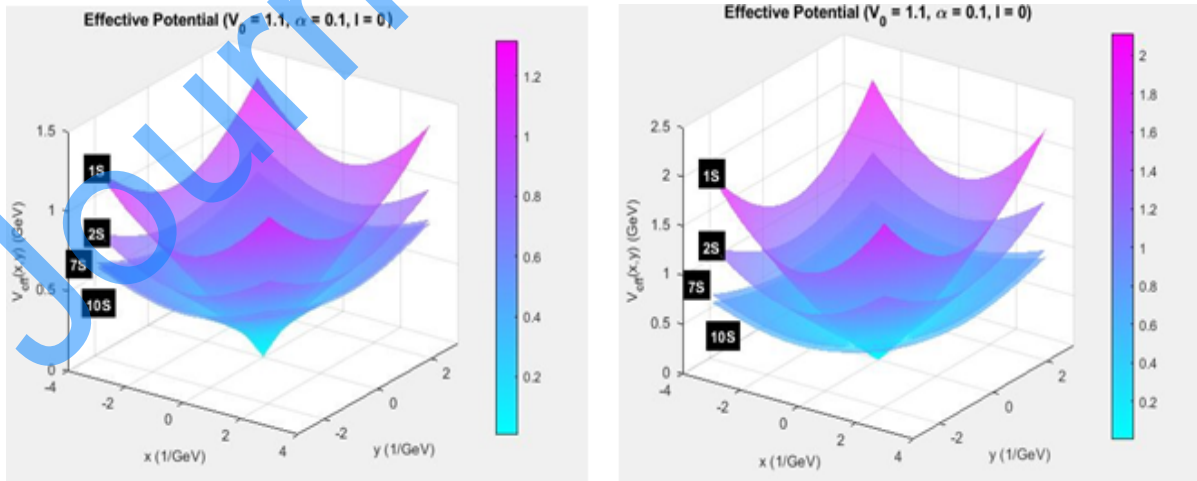
Now, to plot the shape of the effective potential based on the non-RRSE and RRSE, we use the relativistic and nonrelativistic mass spectra and the constituent mass of quarks in 1s: Y(94603) in the

ground state  $n = 0, \ell = 0, n_r = 0$ ,  $2s: Y(10023)$  in the first excited state and two predicted highly excited states  $7s: Y(10860)$  and  $10s: Y(11020)$ . The results of shape, depth, and asymptotic lines of potential are affected by changing the reduced mass of  $b - b^*$  bound state. From the curve and the shape of the effective potential function, we can determine the minimum point where the transition is created and how steep and potential wall behaviors give us the properties of the bound state. As we can see in the relativistic and nonrelativistic plots in Figures 1 and 2, the mass of the bound state changes the curvature of the curve of the potential shape. In the highly bound state mass value (highly resonance states), the shape becomes deeper, which allows us to determine the minimum energy eigenvalue in the mass spectra of the bound states. These properties can help us to understand the bound states' internal dynamics. In high energy physics, properties of potential interaction are an interesting open question for future research at LHC, Super KEKB accelerator (Belle II), ALICE Experiment, LHCb Experiment, CMS, and ATLAS. Therefore, by plotting HCP with different potential parameters, we can explain how sensitive mass spectra and shape of potential in QHO and GBM models and analyze the experimental data. The form of the plot and the depth of the potential well can be used to assign the other hadronic states to be constituted of more than two quarks. Based on QHO and GBM models, one can get a suitable HCP to explain the dynamics of the hadronic bound states by relativistic correction on mass without involving spin interactions. 3D HCP shapes and mass spectra based on non-RRSE and RRSE can represent how sensitive the depth of potential is well to the relativistic mass correction. The results of relativistic correction on the effective potential of Bottomia bound states have been performed in MATLAB R2021a software and Excel 2022 and they are presented in Fig. 1 and Fig. 2.



**Fig. 1.** Effective potential well with the relativistic correction on mass

1s:  $\Upsilon(94603)$ , 2s:  $\Upsilon(10023)$ , 7s:  $\Upsilon(10860)$ , 10s:  $\Upsilon(11020)$



**Fig. 2.** Multiple surfaces of the effective potentials: left (RRSE) and right (non-RRSE) of

1s:  $\Upsilon(94603)$ , 2s:  $\Upsilon(10023)$ , 7s:  $\Upsilon(10860)$ , 10s:  $\Upsilon(11020)$

#### 4. Conclusion

This study concentrates on describing the asymptotic behavior of the polarization loop function in the quantum gauge interaction to define the mass spectrum of bound Bottomia states under the relativistic correction on the mass of constituent particles in high energy physics. Comparing nonrelativistic mass spectra and relativistic correction on mass spectra based on the modified Schrödinger equation, we can conclude that our results may explain and predict resonance states using 3D HCP well's shape of the pair  $b - b^*$  bound states. Based on Table 1, a conclusion of the results which are defined analytically using QHO and GBM models, we may summarize the main concepts as follows:

- 1- The modified Schrödinger equation under the relativistic correction on mass is investigated by applying QHO and GBM models.
- 2- The exponential and linear confinement type of potential is used to determine the mass spectra of Bottomia 1s: Y(94603), 2s: Y(10023), 7s: Y(10860), and 10s: Y(11020).
- 3- The analytical expression of the constituent mass  $\mu_b$  and reduced mass  $\mu = \frac{\mu_b}{2}$  are presented. Results are used for describing mass spectra under relativistic and nonrelativistic formalisms.
- 4- Defined numerical results are compared with the experimentally well-established low-level states 1s: Y(94603), 2s: Y(10023) and high resonance states 7s: Y(10860), and 10s: Y(11020). For these S-wave states, our calculation has the percent differences, computed as

	$\Delta M\%$
<b>1s: Y(94603)</b>	-0.02
<b>2s: Y(10023)</b>	+2.83
<b>7s: Y(10860)</b>	-0.39
<b>10s: Y(11020)</b>	+0.29

In the first excited state, our value is about +2.83 % larger than the experimental one; the reason for this difference can be found in the total Hamiltonian of interactions. In the Bottomia



Hamiltonian of hyperfine (triplet and singlet levels), spin-orbit and tensor interactions are important. On the other hand, the nonperturbed Hamiltonian term that includes relativistic behavior of potential interaction and high temperature conditions can shift the mass spectra to some disagreement with experimental data.

- 5- We used purely static potential, no perturbative loop corrections, and no higher-order momentum-dependent terms  $p_s^{2\tau}$  with  $\tau = 2, 4, \dots$ , hence these terms can justify some disagreement with our results with experimental data. Therefore, as we know, the first radial excited state feels the short-distance Coulombic potential more than the ground state. Since we did not directly include  $\alpha_s$  as a coupling constant, it makes the first radial existing state mass spectrum +2.83 too heavy.
- 6- Using the boundary condition of potential depth  $V_0$ , theoretically calculated and predicted by applying QHO and GBM models, other higher resonance states, such as 11s and 12s. With the relativistic bound state masses 11.1148, 11.1748, and nonrelativistic 10.8758, 10.9195, the constituent mass of the bottom quark 5.0662 and 5.0809 are determined in 11s and 12s states, which have not been confirmed and discovered yet [29,30].

QHO and GBM methods used in this theoretical research can be one of the most concrete works in the calculation and prediction of hadronic bond states. Therefore, theoretical calculations with optimized and modified potentials for the structure of the strong interaction can provide new and different approaches, especially in perspectives in particle physics, as accelerator and LHC technologies continue to expand and become stronger. One of these approaches is discussed in this article. This approach describes the behavior and properties of the bound Bottomia within the relativistic mass correction.

### Acknowledgment

The author would like to acknowledge the use of AIA and tools to improve the grammar and clarity of this manuscript. All intellectual content and conclusions are solely those of the author. We

are indebted to the referees and are very grateful to the referee for the careful reading of the paper and for their comments and detailed suggestions which helped us to improve considerably the manuscript.

### Funding

The theoretical research described here was supported by Foundation grant 1404.

### References

- [1] J. Yoh. The discovery of the  $b$  quark at Fermilab in 1977: The experiment coordinator's story, AIP Conf. Proc., 424 (1998) 29–42. <https://doi.org/10.1063/1.55114>
- [2] D. Asner, H. Atmacan, et al. Belle II Executive Summary, arXiv:2203.10203v2, [hep-ex], (2022). <https://doi.org/10.48550/arXiv.2203.10203>
- [3] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C. Shen, et al. The xyz states: experimental and theoretical status and perspectives, Phys. Rep., 873 (2020) 1-154. <https://doi.org/10.1016/j.physrep.2020.05.001>
- [4] C. Bokade, Bhaghyesh. Predictions for bottomonium from a relativistic screened potential model, arXiv: 2501.03147v1 [hep-ph], (2025). <https://arxiv.org/pdf/2501.03147v1>
- [5] Zh. Tang, et al. Quarkonium spectroscopy in the quark-gluon plasma, arXiv:2502.09044v1 [nucl-th] (2025). <https://doi.org/10.48550/arXiv.2502.09044>
- [6] D., Fujiwara. Rigorous Time Slicing Approach to Feynman Path Integrals, 1st ed. Springer International Publishing, (2017).
- [7] J. Struckmeier. Relativistic generalization of Feynman's path integral based on extended Lagrangians. arXiv:2406.06530v1[quant-ph], (2024). <https://doi.org/10.48550/arXiv.2406.06530>
- [8] G. Semenoff. Quantum Field Theory, 1st ed. Springer International Publishing, (2023).

- [9] M., Didenykh, G., Efimov, G., Ganbolds, N., Nedelko. Oscillator Representation in Quantum Physics (Lecture Notes in Physics Monographs), 1st ed. Springer International Publishing, (1995).
- [10] W. Greiner, S. Schramm, W. Stein. Quantum Chromodynamics, 3rd ed. Springer International Publishing, (2007).
- [11] R., Gould. Quantum Electrodynamics. Electromagnetic Processes, 1st ed. Springer-Verlag International Publishing, (2020).
- [12] S. Albeverio, R. Høegh-Krohn, S. Mazzucchi. Mathematical Theory of Feynman Path Integrals, 1st ed. Springer-Verlag International Publishing, (2008).
- [13] E. Copson. Asymptotic Expansions: The saddle-point method, 1st ed. Cambridge University Press, (2009).
- [14] K. Barley, J. Vega-Guzm, A. Ruffing, S. Suslov. Discovery of the relativistic Schrödinger equation, Journal uspekhi fizicheskikh nauk, Russian Academy of Sciences, 65 (1) (2022) 90 - 103. <https://doi.org/10.3367/ufne.2021.06.039000>
- [15] V. Badalov, S. Badalov. Generalised tanh-shaped hyperbolic potential: klein–gordon equation's bound state solution, Communications in Theoretical Physics, 75 (2023) 075003. <https://doi.org/10.1088/1572-9494/acd441>
- [16] V. Kher, R. Chaturvedi, N. Devlani, and A. K. Rai. Bottomonium spectroscopy using Coulomb plus linear (Cornell) potential, Eur. Phys. J. Plus, 137 (2022) 1. <https://doi.org/10.1140/epjp/s13360-022-02538-5>
- [17] W. Kutzelnigg. Theory of the expansion of wave functions in a Gaussian basis, Quantum chemistry journal, 51 (6) (1994) 447-463. <https://doi.org/10.1002/qua.560510612>
- [18] M., Rushka, J., Freericks. A completely algebraic solution of the simple harmonic oscillator, Am J Phys. 88 (11) (2019) 976–985. <https://doi.org/10.1119/10.0001702>

- [19] J., Kelley, J., Leventhal, “Ladder Operators for the Harmonic Oscillator. Problems in Classical and Quantum Mechanics”, Springer International Publishing, (2017).
- [20] D.H. Lehmer. On the Maxima and Minima of Bernoulli Polynomials, American Mathematical Monthly, 47 (1940) 533–538. <https://doi.org/10.2307/2303833>
- [21] Zhi-Wei Sun, Hao Pan. Identities concerning Bernoulli and Euler polynomials, Acta Arithmetica. 125 (2006) 21–39. <https://doi.org/10.48550/arXiv.math/0409035>
- [22] A. Shafie, J. Naji, et al., Approximation and analytical study of the relativistic confined two particles state within the complex potential in the isotropic medium, Journal of Particle Science and Technology, 9 (2) (2023) 85-101. <https://doi.org/10.22104/jpst.2024.6528.1243>
- [23] A. Ahmadov, K. Abasova, M. Orucova. Bound state solution Schrödinger equation for extended Cornell potential at finite temperature, Advances in High Energy Physics, 1861946 (2021) 1-13. <https://doi.org/10.1155/2021/1861946>
- [24] Particle Data Group. Progress of Theoretical and Experimental Physics, 083C01. (2020)
- [25] V. Badalov, et al. Dirac equation solution with generalized tanh-shaped hyperbolic potential: application to charmonium and bottomonium mass spectra), arXiv:2409.15538v3[hep-ph] (2025). <https://doi.org/10.48550/arXiv.2409.15538>
- [26] M. Abu-Shady. Heavy Quarkonia and  $bc^*$  mesons in the Cornell potential with harmonic oscillator potential in the N-dimensional, International Journal of Applied Mathematics and Theoretical Physics, 2 (2) (2016) 16-20. <https://doi.org/10.11648/j.ijamtp.20160202.11>
- [27] B. Bukor, J. Tekel. On quarkonium masses in 3D non-commutative space, Eur. Phys. J. Plus 138 (2023) 499. <https://doi.org/10.1140/epjp/s13360-023-04049-3>
- [28] K. Purohit, A. Rai. Quarkonium spectroscopy of the linear plus modified Yukawa potential, Physica Scripta, 97 (4) (2022) 044002. <https://doi.org/10.1088/1402-4896/ac5bc2>
- [29] Q. Li, et al. Canonical interpretation of  $Y(10750)$  and  $Y(10860)$  in the  $Y$  family, Eur. Phys. J. C 80 (1) (2020) 1-11. <https://doi.org/10.1140/epjc/s10052-020-7626-2>

- [30] J.Wang, X. Liu. Identifying a characterized energy level structure of highly charmonium well matched to the peak structures in  $e^+e^- \rightarrow \pi^+ D_0 D^*$ , Phys. Lett. B, 849 (2024) 138456.

<https://doi.org/10.48550/arXiv.2306.14695>

Journal Pre-proofs