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Approximation and analytical study of the relativistic confined two particles state within the complex potential in the isotropic medium

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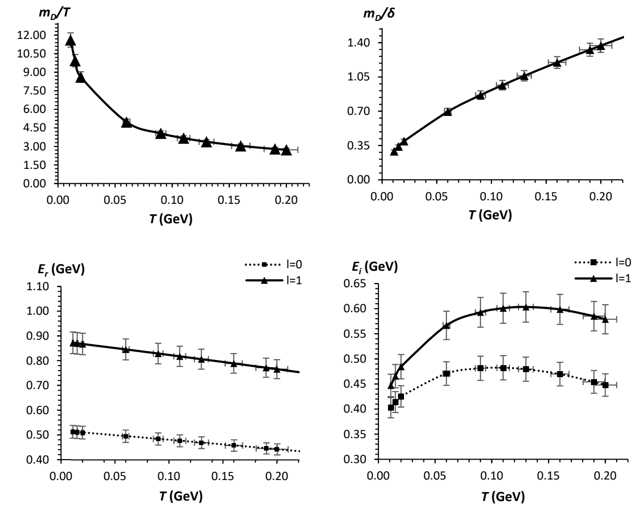
HIGHLIGHTS

- The dissociation of particle-antiparticle confined systems through a complex potential.
- The real component of the confining nonperturbative and relativistic parts of potential.
- Prediction the masses of the charmonium confined state and study the heat dynamics and attributes of the system.

GRAPHICAL ABSTRACT



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ABSTRACT

We have studied the dissociation of particle-antiparticle confined systems through a complex potential, which comes from adjusting and correcting both the perturbative and nonperturbative terms at finite temperatures. The real component of the confining nonperturbative potential makes the bound state stronger; in contrast, the absolute value of the imaginary component contributes more significantly to the thermal width at higher temperatures. The results are presented using the relativistic Bethe-Salpeter equation for the real and imaginary parts of the given potential within the framework of the asymptotic pattern and characteristics of the real and imaginary-time Green's functions of the bound state as charged particles that couple to a gauge field in any external field. The relativistic behavior of interaction within the Sturmian representation for two intertwined spaces was extracted based on the mathematical quantum field theory technique in the perturbative calculation of the quantum field theory. The expectation value of the vacuum of field operators is defined as a sum of operator product expansion methods. The results are applied to predict the masses of the charmonium confined state and study the heat dynamics and attributes of the system. The expectation values agree with the experimental and theoretical data found within scholarly works and among academic researchers.

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1. Introduction

The study of the interaction between charm and anticharm quarks is one of the core topics in particle and high-energy physics. In such studies, the constituent particles' thermal properties and relativistic behavior are essential compared to the other properties [1,2]. The traditional framework of strong interaction is based on particle behavior, and the bound state is described as a point-like system. In this article, we try to present the real and imaginary potential of interaction. In this regard, we do not neglect the time behavior in the complex potential interaction and two-particle imaginary time Green's function [3,4].

As we know, in recent years, hadronic bound states physics has entered a new period with the rise of increased control of the strong interaction colliders such as BABAR collaborations, SIDIS data from COMPASS, ALICE experiment, and exact SIA measurements [5-7]. Hence, many methods and approaches concentrate on real-imaginary functions with thermal properties at finite temperatures. The real-imaginary formalism of potential and Green's function (correlator) are essential and influential in describing the thermal properties of a charmonium confined state, as well as relativistic properties in hadronic high energy physics. In real time, we can calculate Green's function of a bound state using the Feynman path integral, and it should be averaged over fields in the initial and final state. Therefore, the generating functional, which is a critical object in quantum field theory, can be identified with the quantum partition function, and if we evaluate on imaginary time $t = -i\tau$ and trace over the initial and final state, then set initial and final states equal and sum over them, we can find the relation between quantum mechanics and thermodynamic expectation values $t \to -i\tau$, where the imaginary-time variables are $\tau \in [0, \beta]$, $\beta = \frac{1}{k_b T}$, where T is the temperature, and k_b is Boltzmann constant [8].

Partition functions or thermodynamic quantities must be modified appropriately by imposing periodicity conditions on the fields and the Euclidean spacetime integrals. Afterwards, the imaginary time and the real-time Green's function are related by relation $G(\vec{r},t) = G(\vec{r},i\tau)$. This relation shows that we could define Green's function $G(\vec{x}, z)$ in a complex form equal to Green's function of quantum mechanics and thermodynamics, i.e., $G(\vec{r},z)$, $z \in C$ for real z and similar to the thermodynamic average for imaginary z. $G(\vec{r},t)$ computes the real-time correlation function at zero temperature while $G(\vec{r},i\tau)$ computes real-time correlation functions in a quantum field theory at finite temperature [8]. These two correlation functions, $G(\vec{r},i\tau)$ and $G(\vec{r},t)$, are different. We can obtain the correlation function at the $T = 0$ temperature by taking $\lim_{\beta\to\infty} G(\vec{r},i\tau)$, where we know that $G(\vec{r},t) = \lim_{\beta\to\infty} G(\vec{r},i\tau)$. Now we try to apply the main idea of the real and imaginary method of Green's function $G(\vec{r}, z)$ to calculate Green's function in the form of Feynman functional integrals or Feynman path integrals. This idea of Feynman path integrals is defined by the Gaussian-equivalent representation path Integrals approach, which develops various actual problems of modern quantum physics [9].

Using this idea, we should present the influence functional and complex potential of the charmonium confined state in a thermalized environment (isotropic medium) in which they interact. Calculating the confined mass spectra of the charmonium systems with charges and rest masses is one of the important issues in defining the formation behavior and dynamics of hadronic systems at finite temperatures. Our main goal in this article is to treat quantum chromodynamics interactions from a thermal and relativistic point of view and define mass spectra of the charmonium bound states based on the real-time and the imaginary time Green's function and potential interaction. This paper is divided into the following sections. Section 2 explains the real and the imaginary time Green's function and potential interaction for the charmonium bound states. Section 3 calculates the bound state energies of the charmonium bound states in the complex potential. The constituent mass of bounded particles and energy spectrum in the formalism of Feynman path integrals, the complex potential, operator product expansion method, and thermodynamic behavior of the environment (isotropic medium) are calculated in Section 4, and we elaborate the conclusion in Section 5 [10-12].

2. The imaginary-time and the real Green's function

The Green's function of the charmonium bound states in a hot environment (isotropic medium) has been studied in recent years based on the relativistic behaviors of a system. Matsui and Satz have described and suggested such bound states in heavy-ion collision excrement based on ultra-relativistic conditions [13]. Hence, in this research, we study the charmonium bound states at finite temperatures from a theoretical standpoint of the real-imaginary time effect on Green's function and the screened potential of interaction.

First, we present Green's function as Feynman path integral. As we know, too many subjects of particle physics at high energies and ultra-relativistic limits in terms of Feynman path integral are formulated. Our solution is obtained using Gaussian-equivalent representation path integrals in the Feynman functional integral pattern, where the bound state interaction in quantum field theory is described as the interaction functional that depends on the fields $\varphi(r)$. The exact result of functional integral $[V(\varphi)]$ is explained for a limited type of interaction functionals for which the path integral can be simplified to the Gaussian type integral and reduced to the approximate analytical path integral approaches, such as Wentzel–Kramer–Brillouin (also known as the Liouville–Green method), the large N expansion, and the instanton approximation [14-16].

We then developed a universal approach to describe this functional integral for the large, strong coupling regime $g \to \infty$. Subsequently, we used the operator product expansion method of the total Hamiltonian of the charmonium bound state based on the zero-point state of the system. This means that we have to present the functional path integral in the form of the Gaussian equivalent representation (also known as the Gaussian equal representation of functional integrals), which is defined by the real and the imaginary time Green's function and interaction function of potential $[V(\varphi)]$. Using the operator product expansion method [17], we can find the solution of differential equations of correlators for the charmonium bound state in quantum field theory and nonrelativistic quantum mechanics. After that, based on the Bethe-Salpeter equation,

$$H_{tot}\Psi(r) = E_\ell \Psi(r) \Longrightarrow$$

$$\left[\sqrt{m_1^2+p^2}+\sqrt{m_2^2+p^2}+V(r)\right]\Psi(r) = E_\ell\Psi(r) \qquad (1)$$

we can define the energy spectrum of the charmonium in the relativistic conditions under the thermal properties of a system. $H_{tot} = H_{nrel} + H_{rel} + H_s + H_{med}$, where H_{nrel} is the classical Hamiltonian describing the quarks in zero-state (vacuum state), H_{med} is the Hamiltonian of the isotropic medium in which the charmonium propagates, H_s is spin-spin, spin-orbit, and tensor interactions, and H_{rel} represents the nonperturbative relativistic term of interaction and also the interactions between the isotropic medium and the charmonium bound state. Thus, the non-perturbing additives associated with the relativistic nature of the bound state are included in Eq. (1).

We consider the interaction of the charmonium quarks in the neutral field $\varphi(r)$, and an explanation of interaction is given in the Euclidean metrics and Minkowski spacetime. The propagator (Green's function) of the charmonium quarks with the rest masses m_i is $G\,(t, r_1; t, r_2 |\, 0, \acute{r}_1; 0, \acute{r}_2)$ [11,18]. As we know, based on the basic results of equilibrium statistical mechanics, the thermal average of an observable X at a finite temperature T is defined as $\langle X\rangle = tr\,\hat{X}\hat{\rho}$, where the symbol tr is the trace of an operator $\left(tr\,\hat{X}\right) = \sum_i \langle i|\,\hat{X}\,|i\rangle$, i.e., it presents the summation over the expectation values of the $\hat{X}$ over an orthonormal basis set. $\hat{\rho}$ is the density matrix, and it describes thermal distribution over the different eigenstates of the system. Hence, the expectation value of the evaluation of the propagator (Green's functions) in a thermal isotropic medium will be the form of a thermal average as follows [18]:

$$\langle G\,(t, r_1; t, r_2 |\, 0, \acute{r}_1; 0, \acute{r}_2)\rangle =$$
$$\frac{1}{Z} tr\left\{e^{-\beta H_{tot}} J^-(t; r_1, r_2) J^+(0; r_1, \acute{r}_2)\right\} \qquad (2)$$

with $z = tre^{-\beta H_{tot}}, \beta = \frac{1}{k_b T}$.

We know that the considerable time and effective interaction behaviors of the propagator at $t \to \infty$ will be $G\,(t, r_1; t, r_2) \sim e^{-itV(r_1 - r_2)}$. Our goal is to provide a short review of the Feynman path integral method and functional integral technique in quantum field theory and high energy hadronic interactions due to calculating the propagator Eq. (2). The functional integral technique determined the total energy and mass spectrum of charmonium bound state within nonrelativistic quantum mechanics when a complex screened interaction potential is selected. The Bethe-Salpeter Eq. (1) or the relativistic form of the Schrödinger equation provides mathematical techniques for describing bound state formalism with the relativistic corrections; hence, the theoretical challenge is reduced to defining the bound state properties with complex screened interaction potential based on Feynman's functional path integral and within quantum field theory and quantum electrodynamic ideas [11].

The following describes the hadronic screened complex potential of interaction and mass spectrum of the bound state. This idea focuses on thermal and ultra-relativistic corrections within perturbation theory. Since the binding energy and relativistic mass of the charmonium bound state from the Bethe-Salpeter equation are difficult to calculate, we determine these properties using Feynman's functional path integral and the quantum field theory method. The confined charmonium's mass is defined by the real-time and the imaginary time Green's function and potential interaction based on the asymptotic behavior of the correlation function $\Pi(r-r')$ overtime for the charged scalar particle in the fields $\varphi(r)$ [19]. Applying the functional method to the function $\Pi(r-r')$ and then using functional Feynman path integral in nonrelativistic quantum mechanics, we define the asymptotic behavior of bound state wave function at large distances, then by averaging and integrating $\Pi(r-r')$ over the field $\varphi(r)$, one can define the mass spectrum of the charmonium bound state and energy eigenvalue.

Next, we start with the motion of charged particles in the field $\varphi(r)$ with complex screened type potential $V(r)$ and present Green's function $G_m(r,r')$ of potential field $\varphi(r)$. In quantum field theory, a propagator is a mathematical tool used to describe the behavior of particles or fields in terms of correlations between different points $(r \to r')$ in spacetime. It is a solution to Schrödinger equations $[(i\partial_r + g\varphi(r))^2 + m^2]G_m(r|\varphi(r)) = \delta(r-r')$ for a given field in the spacetime points r and r', and with the coupling constant g and specific boundary conditions. Green's functions are used to explain $\Pi(r-r')$, due to calculating S-matrix elements, differential cross-sections, and total cross-sections in high energy physics. Therefore, the behavior of the charged particle with mass m in the field $\varphi(r)$ are presented by the current $J(r) = R^+(r)R^-(r)$, where $R^+(r)$ and $R^-(r)$ are the particle fields, and r represents the spacetime coordinates (x,y,z,t). In the electromagnetic potential field, the general formula for the real current of fermion fields is presented as $J(r) = R^+(r)\gamma_\mu R^-(r)$, where γ_μ is the gamma matrix, μ is the index representing the spacetime direction, $R^+(r)$ is the fermion field, and $R^-(r)$ is the adjoint (conjugate) fermion's field. The average on the background field $\varphi(r)$ (that limits only the lowest order of two dot Gaussian correlator) reads $\langle e^{\{i\int dr\varphi_\alpha(r)J_\alpha(r)\}}\rangle = e^{\{-0.5\int\int dr d\acute{r} J_\alpha(r)D_{\alpha\beta}(r-\acute{r})J_\beta(\acute{r})\}}$.

The Gaussian propagator of the field has the form $D_{\alpha\beta}(r-\acute{r}) = \langle\varphi_\alpha(r)\varphi_\beta(r)\rangle = \delta_{\alpha\beta}D(r-\acute{r}) + \partial_\alpha\partial_\beta\overline{\overline{D}}(r-\acute{r})$, where $D_{\alpha\beta}(r-\acute{r})$ is the covariant function of the random field. The scalar product current of two scalar charged particles with masses m_i, m_j is $\langle J(r)\cdot J(r')\rangle = \langle R^+(r)\cdot R^-(r)|R^+(r')\cdot R^-(r')\rangle$, is the equivalent of $\langle G_{m_i}(r|\varphi(r))\cdot G_{m_j}(r'|\varphi(r'))\rangle$, where $G_{m_i,m_j}(r_i,r_j;r'_i,r'_j) = \langle 0|\hat{T}\varphi(r_i)\varphi(r_j)\varphi(r'_i)\varphi(r'_j)|0\rangle$ is the kernel function of a scalar-charged particle with mass m_i,m_j in the $\varphi(r)$ field and $\hat{T}$ is the time-ordered product of operators. The function $G_{m_i,m_j}(r_i,r_j;r'_i,r'_j)$ can determine the correlators by averaging over the field $\varphi(r)$ and the correlator takes the form:

$$\Pi(r-r') = \langle J(r)J(r')\rangle = \langle G_{m_i}(r|\varphi(r))G_{m_j}(r'|\varphi(r))\rangle = -2i\sum\int\frac{d^3r}{(2\pi)^3}\left[G_{m_i}(r,r'|\varphi(r))G_{m_j}(r,r'|\varphi(r))\right]$$

As we know, the kernel function of a scalar-charged particle with mass m_i,m_j in the $\varphi(r)$ field based on Feynman's functional path integral is:

$$G(r,r'|\varphi(r)) = \int\frac{1}{N}\delta\sigma e^{\{-0.5\int\int dr dr'\varphi(r)G^{-1}(r,r')\varphi(r')\}}e^{-g[V(\varphi)]} = \frac{1}{\tilde{N}}\delta\sigma e^{\{-0.5\int dr(\varphi(r))^2 - g[V(\varphi)]\}}$$

where N and $\breve{N}$ are the normalization scaling coefficient with the conditions $\int d\sigma = 1$, and $\int dr' G^{-1}(r,r')G(r,r') = \delta(r-r')$, $r = (ct, \vec{r}) = (r^0, \vec{r})$, and $\tau = it$. Here τ_i is the proper time of composite particles i.

We assume that at the initial time, composite particles are at rest and interact with each other only through fields. We study the asymptotic behavior $r \to \infty\, (or\ t \to \infty)$ of $\Pi(r-r')$. Using the variational parameters q, s, the functional integral $G(r,r'|\varphi(r))$ is described in the form $G(r,r'|\varphi(r)) \geq \sim e^{\{M\}}$, where M is the bound state mass spectrum $M = min\left\{-0.5\sum_n \left[ln(1+q_n) - \frac{q_n}{1+q_n}\right] - 0.5\sum_n s^2 - \int d\sigma U[\varphi,s]\right\}$. Then, after following the mathematical modifications presented above, the solution to the equation $[(i\partial_r + g\varphi(r))^2 + m^2]G_m(r|\varphi(r)) = \delta(r-r')$ in the functional integral form with the correlator function in the four-dimension spacetime in the path integral form becomes as follows [18,20]:

$$G_m(r,r'|\varphi(r)) = \int_0^\infty \frac{d\mu}{(4\pi\mu)^2} e^{\left\{-\mu m^2 - \frac{(r-r')^2}{4\mu}\right\}}$$

$$\int_0^\infty dB(\xi) e^{\left\{i\int_0^1 d\xi(r-\acute{r}+2\sqrt{\mu})B(\xi)\varphi(r)(rB(\xi)+\acute{r}(1-B(\xi))+2\sqrt{\mu})B(\xi)\right\}} =$$

$$\int_0^\infty \frac{d\mu}{(4\pi\mu)^2} e^{\left\{-\mu m^2 - \frac{(r-r')^2}{4\mu}\right\}}$$

$$\int_0^\infty N\delta B(\xi) e^{\left\{-0.5\int_0^1 d\xi\, (\dot{B}(\xi))^2\right\}} e^{\left\{ig\int_0^1 d\xi \dot{z}(\xi)\varphi(\xi)\right\}} =$$

$$G_m(r,r'|\varphi(r)) = \int_0^\infty \frac{d\mu}{(4\pi\mu)^2} e^{\left\{-\mu m^2 - \frac{(r-r')^2}{4\mu}\right\}}$$

$$e^{\left\{-\min_{(\xi,\sigma,\lambda)>0}\left\{\frac{1}{4\xi}+2\sigma+\frac{\lambda}{2}+g\xi\int_0^\infty d\mu f(k)\right\}((r-r'))\right\}} \tag{3}$$

where

$$z(\xi) = (r-r')\xi + r' - 2\sqrt{\mu}B(\xi)$$

$$B(0) = B(1) = 0$$

$$f(k) = \int e^{-\mu} \int \left(\frac{dk}{2\pi}\right)^4 \breve{D}(k^2)\left[1 - e^{\left\{\frac{ikn\mu}{2\lambda} - \frac{\xi}{6}k^2\left(1-e^{-\frac{\sigma\mu}{\lambda}}\right)\right\}}\right]$$

As we know, the evolution of quantum systems in nonrelativistic quantum mechanics can be alternately provided by the Feynman path integral formulation (FPF). The FPF expresses the transition amplitude between initial and final states in terms of a sum over all possible paths a particle or system can take. The form of FPF in nonrelativistic quantum system can be written as $\langle a_f(t)|a_i(t)\rangle = \int_0^\infty D\left[a(t)e^{\frac{i}{\hbar}S[a(t)]}\right]$, where $S[a(t)]$ is the action, $\langle a_f(t)|a_i(t)\rangle$ is the transition amplitude between the initial position $a_i(t)$, and the final position $a_f(t)$, i.e., the integration over all possible paths $a(t)$ that the particle or system can take between i and f positions. FPF allows one to consider all possible states and paths in classical and quantum physics. Hence, the contributions of each path can lead to the quantum mechanical behavior of the system. Therefore, we use the evaluating form of FPF and note that the FPF is a powerful tool in quantum mechanics, and it has many applications in various fields, including condensed matter physics, quantum field theory, and quantum statistical mechanics. Therefore, using Eq. (3), we determine the polarization function as follows [18]:

$$\Pi(r-r') = \int_0^\infty \int_0^\infty \frac{d\mu_i d\mu_j}{(8\pi^2\mu_i\mu_j|r-r'|)^2} \Omega(\mu_i,\mu_j)$$

$$e^{\left\{-\frac{|r-r'|}{2}\left[\left(\mu_i m_i^2 + \frac{1}{\mu_i}\right) + \left(\mu_j m_j^2 + \frac{1}{\mu_j}\right)\right]\right\}}{}_{i,j=1,2} \tag{4}$$

where parameters μ_i, μ_j is considered the mass component in the connected condition. This mass differs from m_i, m_j, the masses of a free condition (rest mass). When determining the mass spectrum of a relativistic confined system, the constituent mass of the component, which differs from the mass of the source particle (i.e., the initial particle), is usually introduced. Specifically mentioned, at the exposition of a mass spectrum of confined particles such as hadrons consisting of quarks, the mass of valent and current quarks is usually introduced with variations that make them unique from one another. Then, $\Omega(\mu_i,\mu_j)$ has the form [18]:

$$\Omega(\mu_i,\mu_j) =$$

$$c_i c_j \int\int \delta r_i\, \delta r_j e^{\left\{-0.5\int_0^t d\tau\left(\mu_i \vec{\dot{r}}_i^{\,2}(\tau) + \mu_j \vec{\dot{r}}_j^{\,2}(\tau)\right)\right\}} e^{\{U_{ij}\}} \tag{5}$$

where c_i, c_j are normalization constant during the presentation of the Green function in the FPF, where we can present these normalizations in the form of

$1 = c\int_0^{\nu} d\nu\, e^{\int_0^{t} d\tau \frac{mv^2}{2}}$ (for details, see [26]), and the potential of interactions included in the last term, and reads:

$$U_{ij} = \frac{(-1)^{i+j}g^2}{2}\int_0^t\int_0^t d\tau_1 d\tau_2\, \dot{Z}_\alpha(\tau_1) D_{\alpha\beta}\left(Z^i(\tau_1) - Z^j(\tau_2)\right)\dot{Z}_\beta(\tau_2) = \frac{(-1)^{i+j}g^2}{2}\int_0^t\int_0^t d\tau_1 d\tau_2 \left(\frac{\vec{r}}{|r|} + \frac{\vec{r}_i(\tau)}{c}\right)\left(\frac{\vec{r}}{|r|} + \frac{\vec{r}_j(\tau)}{c}\right)\int\frac{dq}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{ds}{2\pi} D_{\alpha\beta}\left(\vec{q}^2 + \frac{s^2}{c^2}\right) \times \times e^{\left\{is\tau + \frac{is}{c}\left(r_i^0(\tau_1) - r_j^0(\tau_2)\right) + i\vec{q}\left(\vec{r}_i(\tau_1) - \vec{r}_j(\tau_2)\right)\right\}} = U_{ij}{}^1 + U_{ij}{}^2 \quad (6)$$

In the term $U_{ij}{}^1 = -U_{11}{}^1 + 2U_{12}{}^1 - U_{22}{}^1 = \int_0^t d\tau\left(U(r) + \int\alpha\frac{dk}{(2\pi)^2 k^2}\right)$, the $U_{12}{}^1$ corresponds to the contribution of one gluon exchange, $U_{11}{}^1, U_{22}{}^1$ where ($i = j$) (determines a renormalization of a mass operator, the term $U_{ij}{}^2$ gives only cross term, and addend is equal to zero. $U(0) = \int\alpha G_F\frac{dk}{(2\pi)^2k^2}$ corresponds to the renormalization of the mass operator in nonrelativistic interaction behavior, parameter c is the speed of light, and α is the coupling constant of one gluon exchange.

We suppose that in the initial moment, quarks are in the rest state and interact with each other only by colored field. For this specific case, we have $\Theta_{ij} = 1 + \frac{\vec{r}}{|r|c}\left(\vec{r}_i(\tau_1) - \vec{r}_j(\tau_2)\right) + \frac{1}{c^2}\left(\vec{r}_i(\tau_1)\vec{r}_j(\tau_2)\right) = 1$. From $r^\mu = (ct, \vec{r})$, $\tau \equiv it$, $\vec{r} = \vec{r}_i(\tau_1) - \vec{r}_j(\tau_2)$, it follows that with the asymptotic behavior of U_{ij} must depend on t in a linear way [18]. According to this, the dependence of r^0 is Euclidean time, and τ is the proper time of the composite particle, chosen as $r^0 = r_1^0(\tau_1) - r_2^0(\tau_2) = c(\tau_1 - \tau_2) = c\tau$, where τ_1 and τ_2 are considered the proper adequate time for composite particles 1 (charm quark) and 2 (anticharm quark), respectively. The functional $\Omega(\mu_i,\mu_j)$ contains U_{ij} potential and nonpotential interaction within the four-dimension spacetime in the Feynman path integral form, which resembles the behavior of two particles with masses μ_i , μ_j in nonrelativistic quantum mechanics. Hence, based on the above presentation of the propagator at $|r - r'| \to 0$, the coupled state with the total mass M of bounded particles with the rest masses m_i , m_j is defined as the following $G\left(r,r'\middle|\varphi(r)\right) = \frac{const}{|r-r'|}e^{\{-M|r-r'|\}}$. Therefore, the mass spectrum which depends on the coupling constant of strong interaction, satisfies the relation $g \gg 1$, with $M = (1.022g)^{0.25}\left(\int\left(\frac{dk}{2\pi}\right)^4\tilde{D}(k^2)\,k^2\right)^{0.25}$. The functional $\Omega(\mu_i,\mu_j)$, which contain potential and nonpotential interactions at $|r - r'| \to 0$, has the form $\Omega(\mu_i,\mu_j) \cong e^{\{-|r-r'|E_n(\mu_i,\mu_j)\}}$, where $E_\ell(\mu_i,\mu_j)$ is an eigenvalue of the Schrödinger equation $H_{tot}\Psi = E_\ell(\mu_i,\mu_j)\Psi$ of two bounded particles with the total Hamiltonian H_{tot}. Parameters m_i, m_j are the current masses of interacting particles and μ_i, μ_j are the constituent masses of interacting particles, i.e., the mass of particles inside the coupled state. Considering the above equations, we can explain that in QFT, the bound state of two scalar particles can be described using the polarization function, which is closely related to the propagator or Green's function of the scalar field. Hence, we can define the $\Pi(r - r') \cong e^{\{-M|r-r'|\}}$, at $|r - r'| \to 0$. Therefore, the mass spectrum of charmonium bound state reads $M = -\lim_{|r-r'|\to\infty}\frac{1}{|r-r'|}\ln\Pi(r - r')$. The mass of the confined state is determined based on the equation: $\mu_i^2 + 2\mu_i^2\left(\min_{\mu_i} E_\ell\right) = m_i^2$, and the steepest descent method is characterized as below:

$$M = \min_{\mu_i,\mu_j}\left(\frac{m_i^2}{2\mu_i} + \frac{m_j^2}{2\mu_j} + \frac{\mu_i + \mu_j}{2} + E_\ell\right) \quad (7)$$

The temperature dependence of the gluon-exchange Coulombic term and string term in the charmonium bound state in the hot medium is typically incorporated through the Debye mass (m_D) at the finite critical temperature T_c, here we used $T_c \approx 220\,MeV$. The critical temperature (the dissociation or melting temperature) represents the temperature at which charmonium bound states are significantly affected and start to separate into their constituent parts in a hot medium. The precise value of the critical temperature for the charmonium bound state depends on various factors, including the specifics of the theoretical model or approach used to describe the system. The specific techniques used to describe the bound states' temperature dependence depend on the theoretical methods and models.

Here are a few commonly used formulas to calculate the thermal behavior and thermal effect on the system. Using Eq. (7), we explain that the bound state with a mass M at the finite temperature in a hot medium: 1) can create or can be stable if $M < \infty$ and $M \neq \sum_{i=1}^{n} \mu_i$, 2) the bound state cannot create, or it cannot be stable, and the scalar particles exist as two self-contained and unconnected states if $M < \sum_{i=1}^{n} \mu_i$. Hence, we can present the critical mass value $M_c = \sum_{i=1}^{n} \mu_i$.

3. The real and imaginary interaction potential

Now, we will discuss the charmonium states closer to the hot thermal isotropic medium surrounded by photons, gluons, light quarks, and other fermions. We choose the Yukawa potential (the screened Coulomb potential) and suppose that the temperature of isotropic medium $T \prec T_c$; hence, the charmonium bound can exist under this condition, and quarks can be confined. The corresponding total Hamiltonian $H = H_{np} + H_p$ describes the system. H_{np} is the Hamiltonian characterizing the charmonium quarks in the vacuum state and relativistic behavior of interactions, $H_{np} = H_{nrel} + H_{med} + H_{rel} + H_i + H_f$ is the interaction Hamiltonian between the charmonium quarks and particles of the thermal isotropic medium, and the relativistic interactions, imaginary potential, and field interactions (we choose all interaction terms that do not contain a quadratic form of canonical variables based on normal ordering methods). Now, based on the Matsubara propagator and $D_{\alpha\beta\lfloor\alpha=\beta=0} = \frac{-1}{p^2+m_D^2} + i\frac{\pi m_D^2 T}{|p|(p^2+m_D^2)^2}$, the real-time boson propagator [10], the definition of the green's function $G(t, r_1; t, r_2) \sim e^{-itU(r_1-r_2)}$, $r = r_1 - r_2$ within the Coulomb-Yukawa type potential with the real part of Coulomb term $\left(-\frac{g^2}{4\pi}(m_D + \frac{e^{-m_D r}}{r})\right)$ and the string term potential $\left(2\sigma(\frac{1}{m_D} + \frac{e^{-m_D r}-1}{r m_D^2})\right)$, and also with imaginary part $\Xi(m_D T)$ at asymptotic limit $t \to \infty$ reads:

$$G(t, r_1, r_2) = e^{\left\{\frac{i}{2}\int\int dr_1 dr_2 J_\alpha(r_1) D_{\alpha\beta}(\acute{r}_1 - r_2) J_\beta(r_2)\right\}}\lfloor_{\alpha=\beta=0} \sim e^{it\left\{-\frac{g^2}{4\pi}\left[m_D + \frac{e^{-m_D r}}{r}\right] + 2\sigma\left[\frac{1}{m_D} + \frac{e^{-m_D r}-1}{r m_D^2}\right]\right\}} e^{\{\Xi(m_D T)t\}} \tag{8}$$

where g is a magnitude scaling constant, i.e., is the amplitude of potential, m_D is the Debye screening mass, and $\Xi(m_D T)$ is the imaginary part of the interaction function that describes the damping of the in-isotropic medium charmonium bound state's propagator at long (with σ coupling constant) [21] and short distances (with α coupling constant) [10]. Hence it becomes ($a = p / m_D$):

$$\Xi(m_D r) = \Xi_{short}(m_D r) + \Xi_{long}(m_D T) = -\frac{g^2 T}{4\pi}\theta(m_D T) + \frac{2\sigma T}{m_D^2}\,\xi(m_D T) = -\frac{g^2 T}{2\pi}\int_0^\infty \frac{a da}{(a^2+1)^2}\left[1 - \frac{\sin(a m_D r)}{a m_D r}\right] + \frac{4\sigma T}{m_D^2}\int_0^\infty \frac{da}{a(a^2+1)^2}\left[1 - \frac{\sin(a m_D r)}{a m_D r}\right] \tag{9}$$

where p is the four-momentum of the scalar particle [21]. After carrying out the integration over parameter a, the contribution arising from the short-distance term to the imaginary part becomes (for more details, see Appendix A):

$$\Xi(m_D r) = -\frac{g^2 T}{2\pi}\left(\frac{\pi^2}{8} e^{-|m_D r|}\right) + \frac{4\sigma T}{m_D^2}\left(\frac{\pi^2}{16} e^{-|m_D r|}\right) \tag{10}$$

As we described in the previous section, using Eq. (6), one obtains the potential of interaction corresponding to the isotropic medium modifications to the electrostatics, and the string term becomes as follows:

$$U(r) = \frac{g^2}{(2\pi)^3}\int dq\left(1 - e^{ip\cdot r}\right)\left[\frac{1}{p^2+m_D^2} - i\frac{\pi m_D^2 T}{|p|(p^2+m_D^2)^2}\right] = -\frac{g^2}{4\pi}\left[m_D + \frac{e^{-m_D r}}{r}\right] + 2\sigma\left[\frac{1}{m_D} + \frac{e^{-m_D r}-1}{r m_D^2}\right] + i\Xi(m_D T) \tag{11}$$

The function $\Xi(m_D T)$ for $r = 0$ vanishes and $r \to +\infty$ increases monotonously, and for the charmonium bound state at $r \to +\infty$, $\lim \Xi(m_D T) = 1$. The damping effect in Eq. (6) is characterized by the damping function $\Xi(m_D T)$ and damping factor $\gamma = \frac{g^2 T}{4\pi}\Xi(m_D T)$, and is one of the primary behaviors and effects of the thermal environment (isotropic medium) on charmonium bound states. The thermal environment (isotropic medium) can decrease charmonium bound state amplitude, leading to energy loss in their motion and reducing charmonium bound state resonance and oscillations. The charmonium bound state in a thermal isotropic medium, such as a quark-gluon plasma, based on isotropic medium temperature above or below $T_c = (150 - 220)\,MeV$, can be confined or deconfined. Hence, the properties of charmonium states can be modified due to interactions with the surrounding particles, such as light fermions and gluons, under thermal conditions. The minimum temperature for charmonium states is expected to be $T_{min} \approx (180 - 240)\,MeV$. Therefore, one of the effects of the thermal environment (isotropic medium) on quarkonium states is the damping of their oscillations, which is characterized by a damping factor γ. The γ is a quantity that describes the rate at which the fluctuations and oscillations of the charmonium bound state decay due to Hamiltonian interactions $H_{int-rel}$ with the thermal environment. The Hamiltonian interactions with the thermal isotropic medium are related to the imaginary part of the energy of the charmonium state, which describes the decay width, lifetime, damping rate, scattering rate, and spectral properties [10]. Hence, the Hamiltonian interactions consist of interactions with the surrounding particles in an isotropic medium and relativistic interactions of constituent particles in the bound states.

In this research, we consider a thermal environment with the condition $T < T_c$, where we use $T_c \approx 200\,MeV$. Therefore, the charmonium interaction in an isotropic medium with an effective potential can lead to a stable bound state. The effective potential for quarkonium states in a thermal isotropic medium includes both the imaginary and the real parts of the potential interaction. The effective potential $U(r)$ refers to the potential energy felt by the charm and anticharm quarks (the constituent particles) that form the charmonium bound state. The effective potential $U(r)$ can be mediated by strong interaction, strong nuclear energy, or external electromagnetic energy. It plays an important role in describing the behavior of the confined state. Hence, the Hamiltonian reads:

$$H_{int\text{-}rel} = H_0 + H_i + H_{med} + H_{rel} + H_{LS} + H_{SS} + H_T + H_f \qquad (12)$$

where H_0 is nonrelativistic Hamiltonian, H_{LS} is spin-orbit interactions, H_{SS} is spin-spin interactions, H_T is tensor interactions, and H_{rel} is the nonperturbative relativistic interaction. The Eq. (12) is defined by:

$H_{rel} = U(r)\left[\left(1 + \frac{\ell(\ell+1)}{(2\hbar c r\mu)^2}\right)^{-\frac{1}{2}} - 1\right]$, where $U(r)$ is one gluon exchange potential and H_i is the imaginary potential in the strong interaction. The imaginary potential arises from the absorption or emission of bosons by the quark in the bound states (constituent quarks), which leads to the creation of quarkonium and is related to the decay width and self-coupling energy of the state. The Hamiltonian provides insights into the properties of the quark-antiquark confined conditions in dense and high-thermal energy conditions (isotropic medium), such as quark-gluon plasma and the core of stars. Given this, we have received the nonperturbative contribution to the Hamiltonian of interaction, which is connected to the relativistic aspect of a confined system [22,23]. As we know, the real term of Yukawa-type potential with the scalar and vector terms in the hot isotropic medium is given by [24,25].

$$U(r) = -\frac{g^2}{4\pi}\left[m_D + \frac{e^{-m_D r}}{r}\right] + 2\sigma\left[\frac{1}{m_D} + \frac{e^{-m_D r} - 1}{r m_D^2}\right] = U_v(r) + U_s(r) \qquad (13)$$

where $U_s(r) = \varepsilon U_v(r), -1 \preccurlyeq \varepsilon \preccurlyeq 1$, $V_0 = \alpha Z$, Z is the atomic number, α is the fine structure constant, and $\varepsilon = \frac{U_s(r)}{U_v(r)}$ is a constant. Hence, the total Hamiltonian of the charmonium bound state within the hot isotropic medium reads:

$$H_{tot} = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} - \frac{g^2}{4\pi}\left[m_D + \frac{e^{-m_D r}}{r}\right] + 2\sigma\left[\frac{1}{m_D} + \frac{e^{-m_D r} - 1}{r m_D^2}\right] + i\Xi(m_D T) - U_v\left[\frac{1}{\sqrt{1 + \frac{\ell(\ell+1)}{r^2 c^2}}} - 1\right] + \frac{1}{r}\left[\left\{\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} + \frac{4}{\mu_1\mu_2}\right\}(LS)_+ + \left\{\frac{1}{\mu_1^2} - \frac{1}{\mu_2^2}\right\}(LS)_-\right]\frac{\partial U_v(r)}{\partial r} - \frac{1}{4r}\left[\left\{\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\right\}(LS)_+ + \left\{\frac{1}{\mu_1^2} - \frac{1}{\mu_2^2}\right\}(LS)_-\right]\frac{\partial U_s(r)}{\partial r} + \frac{1}{12\mu_1\mu_2}\left[(S_1S_2)\Delta U_v(r)\right] + \frac{1}{12\mu_1\mu_2}\left[\frac{1}{r}\frac{\partial U_v(r)}{\partial r} - \frac{\partial^2 U_S(r)}{\partial^2 r}\right] S_{12} \qquad (14)$$

where μ_1 and μ_2 are the constituent mass of quarks, S_1, S_2 are the spin of quarks, and $S_{\pm} = S_1 \pm S_2$, $j = \ell + S$,

$$S_{12} = \frac{4}{(2\ell+3)(2\ell-1)}\left(L^2S^2 - \frac{3}{2}(LS) - 3(LS)^2\right)$$

$$(LS) = \frac{1}{2}\left(j(j+1) - S(S+1) - \ell(\ell+1)\right)$$

$$(S_1S_2) = \frac{1}{2}\left(S(S+1) - S_1(S_1+1) - S_2(S_2+1)\right)$$

Thus, we have obtained all the nonperturbative components to the total Hamiltonian of charmonium interaction in the hot isotropic medium below the critical temperature $T < T_c$ of deconfinement. As presented in Eqs. (7) and (10), we can find the magnitude of the effect of the charmonium bound state's transmission and also determine the behavior of the damping factor γ using an imaginary potential term in a hot isotropic medium. These properties have been connected with the relativistic nature of a system in a hot isotropic medium. We have explained the time relation between Euclidean and our own time charmonium and defined additional associations of interaction. It is clear that the nonperturbative factor is connected to driving quarks rather than each other; therefore, in a nonrelativistic limit, this interaction is absent. Thus, the nonperturbative factor and imaginary term of the potential part are connected to the relativistic nature of a system and hot isotropic medium.

4. Real and imaginary energy eigenvalue in a thermal medium

We now consider the Bethe-Salpeter equation, or Eq. (1), in the fourth order of momentum for a bound state of two particles m_1, m_2 in a potential field $V(r)$ transformed into a canonical variable at a higher order by the method of conversion of symplectic space. When scientifically studying bound states, the mechanism of quantum mechanics and quantum field theory, relativistic modifications, and corrections are of great interest and importance. The Bethe-Salpeter equation is used to explain the bound states interaction of quantum systems. This equation mainly describes relativistic processes and the behavior of interactions. Relativistic effects are accounted for by including relativistic energy and expanding the term $\sqrt{m^2+p_r^2}$ in powers of $\frac{p_r^2}{m}$ as follows:

$$\sqrt{m^2+p_r^2} \approx m + \frac{p_r^2}{2m} - \frac{p_r^2.p_r^2}{8m} + \cdots = m - \frac{1}{2m}\Delta_r + \frac{1}{8m}\Delta_r \cdot \Delta_r = m - \frac{1}{2m}\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}\right) + \frac{1}{8m}\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}\right) \quad (15)$$

then reducing the series to the higher order contribution (fourth and sixth), which allows solving the Bethe-Salpeter equation using the condition:

$$\varepsilon_0(E) = \min_{\omega,\rho} \varepsilon_0(E,\omega,\rho) = (H_{tot} + U_{tot} - E)\Psi(r) = 0 \quad (16)$$

where $\varepsilon_0(E)$ is the ground state energy of the bound state systems based on the modified Schrödinger equation. Hence, the modified Bethe-Salpeter equation based on the expanding powers of $\frac{p_r}{m}$ is known as the relativistic Schrödinger equation. This modification defines the form of wavefunction, mass spectra, and eigenvalue of ground and excited energies.

Now, we can evaluate and compute the relativistic improvements for hadronic bound states to higher order using the relativistic Schrödinger equation to convert symplectic space and perturbation techniques. The symplectic space presents and links the methods of quantum theory and quantum field theory of bound states. The quantized field, being a set of an infinite number of oscillators, describes and models ground or vacuum states. These oscillators retain their oscillating character in quantum field theory. Studies have demonstrated that in quantum mechanics theory, the wavefunction of bound states in interaction fields differs from the Gaussian characteristics. Hence, to apply the ideas of quantum field theory into practice and to solve bound states problems, the canonical variables in Eq. (1) should be converted to symplectic space. This transformation gives us a new equation that exhibits Gaussian behavior and a Gaussian solution for bound state interactions. In this paragraph, according to the interaction conditions, we do not consider the interaction very strong. We ignore the relativistic masses of particles μ_i, and enter the rest mass of the particles m_i, then Eq. (1) reads:

$$\varepsilon_0(E) = \Bigg(\frac{1}{2\mu}p_r^2 - \frac{1}{8\eta}p_r^2.p_r^2 - \frac{g^2}{4\pi}\left[m_D + \frac{e^{-m_D r}}{r}\right] + 2\sigma\left[\frac{1}{m_D} + \frac{e^{-m_D r}-1}{r m_D^2}\right] + i\Xi(m_D T) - U_v\left[\frac{1}{\sqrt{1+\frac{\ell(\ell+1)}{r^2c^2}}} - 1\right] + \frac{1}{r}\left[\left\{\frac{1}{m_1^2} + \frac{1}{m_2^2} + \frac{4}{m_1 m_2}\right\}(LS)_+ + \left\{\frac{1}{m_1^2} - \frac{1}{m_2^2}\right\}(LS)_-\right]\frac{\partial U_v(r)}{\partial r} - \frac{1}{4r}\left[\left\{\frac{1}{m_1^2} + \frac{1}{m_2^2}\right\}(LS)_+ + \left\{\frac{1}{m_1^2} - \frac{1}{m_2^2}\right\}(LS)_-\right]\frac{\partial U_s(r)}{\partial r} + \frac{1}{12 m_1 m_2}[(S_1 S_2)\Delta U_v(r)] + \frac{1}{12 m_1 m_2}\left[\frac{1}{r}\frac{\partial U_v(r)}{\partial r} - \frac{\partial^2 U_s(r)}{\partial r^2}\right]S_{12} - E\Bigg)\Psi(r) = 0 \quad (17)$$

where $\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2}$, $\frac{1}{\eta}=\frac{1}{m_1^3}+\frac{1}{m_2^3}$. Then, we use the form of $r=q^{2\rho}, p_r^2=\frac{q^{2-4\rho}}{4\rho^2}p_q^2, \Psi(r)=q^{2\rho\ell}\Phi(q^{2\rho\ell})$ as a symplectic transformation of canonical variables from $R^n\ to\ R^d$: $\Delta_r=\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}\rightarrow\ \Delta_q=\frac{d^2}{dq^2}+\frac{d-1}{q}\frac{d}{dq}$. Variable $q^{2\rho}$ is a vector in the new symplectic d-dimensional space, where $d=2+2\rho+4\rho\ell$. So Eq. (17) is defined as follows:

$$\varepsilon_0(E)=\left(\frac{1}{2\mu}\frac{q^{2-4\rho}}{4\rho^2}p_q^{2n}-\frac{1}{8\eta}\frac{q^{2-4\rho}}{(4\rho^2)^2}p_q^{2n}.\left(\frac{1}{q^{4\rho-2}}p_q^{2n}\right)-\frac{g^2}{4\pi}\left[m_D+\frac{e^{-m_Dq^{2\rho}}}{q^{2\rho}}\right]+2\sigma\left[\frac{1}{m_D}+\frac{e^{-m_Dq^{2\rho}}-1}{m_D^2q^{2\rho}}\right]+i\Xi(m_DT)+U_{all}-E\right)\Psi(q^{2\rho})=0 \quad (18)$$

Now, based on the normal ordering method, we can define new canonical variables p_q^2 and q^{2n} using the creation ($\hat{a}^\dagger=\frac{1}{\sqrt{2m\omega}}(m\omega\hat{q}-i\hat{p})=\frac{1}{\sqrt{2m\omega}}\left(m\omega\hat{q}-\frac{\partial}{\partial\hat{q}}\right)$) and the annihilation ($\hat{a}=\frac{1}{\sqrt{2m\omega}}(m\omega\hat{q}+i\hat{p})=\frac{1}{\sqrt{2m\omega}}\left(m\omega\hat{q}+\frac{\partial}{\partial\hat{q}}\right)$) operators of the quantum harmonic oscillator system, this quantum system has the frequency ω and these operators in the natural unit ($\hbar=c=1$) are described as follows:

$$\hat{q}^{2n}=\omega^{-n}\frac{\Gamma(\frac{d}{2}+n)}{\Gamma(\frac{d}{2})}+n\omega^{1-n}\frac{\Gamma(\frac{d}{2}+n)}{(\frac{d}{2}+1)}:\hat{q}^2:+\zeta(\hat{q}^2)$$

$$p_q^{2n}=\omega^{n}\frac{\Gamma(\frac{d}{2}+n)}{\Gamma(\frac{d}{2})}+n\omega^{n-1}\frac{\Gamma(\frac{d}{2}+n)}{(\frac{d}{2}+1)}:p_q^2:+\zeta(\hat{p}^2)$$

where $\zeta(\hat{q}^2)$ and $\zeta(\hat{p}^2)$ are the illustration of the total Hamiltonian in the normal ordering form. Then Eq. (18) for the Yukawa type potential becomes:

$$\varepsilon_0(E)=\left(\frac{1}{2\mu}\frac{q^{2-4\rho}}{4\rho^2}p_q^{2}-\frac{1}{8\eta}\frac{q^{2-4\rho}}{(4\rho^2)^2}p_q^{2}.\left(\frac{1}{q^{4\rho-2}}p_q^{2}\right)-\frac{g^2}{4\pi}\left[m_D+\frac{e^{-m_Dq^{2\rho}}}{q^{2\rho}}\right]+2\sigma\left[\frac{1}{m_D}+\frac{e^{-m_Dq^{2\rho}}-1}{m_D^2q^{2\rho}}\right]+i\Xi(m_DT)+U_{all}-E\right)\Psi(q^{2\rho})=0 \quad (19)$$

As described above, $\varepsilon_0(E)$ is the free oscillator Hamiltonian within the potential interaction, i.e., $H=H_0+H_i+\varepsilon_0(E)$. According to the normal ordering method in a quantum oscillating system and the projective unitary representation of the symplectic group, the interaction Hamiltonian H_i in Eq. (12) contains all non-square parts of canonical variables. Therefore, we determined the Hamiltonian with $H_0=\omega(\hat{a}^+\hat{a}^-)$ is the energy of the pure oscillator, H_i is the energy the interaction between the Hamiltonian, and ε_0 is the minimum energy of the bound state system. Relation $\varepsilon_0(E)=min\int\Psi_0^*\hat{H}\Psi_0dq=0$ determines the energy levels in the zeroth approximation based on the projective unitary representation. Eq. (19) shows the expression for the ground state energy ε_0 and it contains of the contribution potential field interaction ε_f, spin-orbit interactions ε_{LS}, spin-spin interactions ε_{SS}, tensor interactions ε_T, and relativistic interactions ε_{rel}, as well as all other perturbative interactions and reads $\varepsilon_0(E)=\varepsilon_0++\varepsilon_f+\varepsilon_{LS}+\varepsilon_{SS}+\varepsilon_T+\varepsilon_{rel}+\cdots$.

Now, we try to define the mass spectra of the charmonium bound state in the Yukawa type potential with the real and imaginary terms within the hot isotropic medium in the ground state $\ell=0$. For simplicity, we neglect $p_q^2.p_q^2$ and all relativistic and spin interactions U_{all}. Next, using lower-order of approximation $e^{-m_Dr}=\sum_{i=0}^{\infty}\left(\frac{(-m_Dr)^i}{i!}\right)=1-m_Dr+\frac{1}{2}m_D^2r^2+\cdots$ and based on the described relation in Appendix A, Eq. (19) is presented as follows:

$$\varepsilon_0(E)=\frac{d\omega}{4}+4\mu\rho^2q^{4\rho-2}\left(-\frac{g^2}{4\pi}\left[m_D+\frac{1}{q^{2\rho}}\left(1-m_Dq^{2\rho}+\frac{m_D^2q^{4\rho}}{2}\right)\right]+2\sigma\left[\frac{1}{m_D}-\frac{1}{m_D^2q^{2\rho}}+\frac{1}{m_D^2q^{2\rho}}\left(1-m_Dq^{2\rho}+\frac{m_D^2q^{4\rho}}{2}\right)\right]+i\left[-\frac{g^2\pi T}{16}(1-m_Dq^{2\rho})+\frac{\sigma\pi^2T}{4m_D^2}(1-m_Dq^{2\rho})\right]-E\right)\Psi(q^{2\rho})=0 \quad (20)$$

Then after simplifying all terms, we consider the ground state of systems by examining the conditions of the normal ordering method (oscillator representation method) and using conditions $E(\omega,\mu,\rho)=\underset{\omega}{min}\,\varepsilon_0(E)$, $\omega=\underset{\omega}{min}\frac{dE(\omega,\mu,\rho)}{d\omega}$, and $\rho\cong\underset{\rho}{min}E(\omega,\mu,\rho)$, the energy eigenvalue E, oscillator frequency ω, and variation parameter ρ of the charmonium bound state become [26]:

$$E=\frac{\Gamma(2+2\rho+2\rho\ell)}{8\mu\rho^2\Gamma(3\rho+2\rho\ell)}\omega^{2\rho}-\frac{g^2}{4\pi}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\omega^{\rho}-\left[\frac{g^2}{8\pi}m_D^2-\sigma\right]\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\omega^{-\rho}+i\left[\frac{g^2\pi m_DT}{16}-\frac{\sigma\pi^2T}{4m_D}\right]\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\omega^{-\rho}+i\left[-\frac{g^2\pi T}{16}+\frac{\sigma\pi^2T}{4m_D^2}\right]=E_r+iE_i \quad (21)$$

and using $\omega = \underset{\omega}{min} E(\omega,\mu,\rho)$ and minimizing the equation ($z = \omega^{\frac{1}{2}}$):

$$\frac{\Gamma(2+2\rho+2\rho\ell)}{4\mu\rho^2}\omega^{3\rho} - \frac{g^2\Gamma(2\rho+2\rho\ell)}{4\pi}\omega^{2\rho} - \left[\frac{g^2}{8\pi}m_D^2 - \sigma\right]\Gamma(4\rho + 2\rho\ell) - i\left[\frac{g^2\pi m_D T}{16} - \frac{\sigma\pi^2 T}{m_D}\right]\Gamma(4\rho + 2\rho\ell) = 0 \quad (22)$$

and then the frequency of the charmonium bound state becomes $\omega^{\rho} = \frac{2g^2\mu\rho^2}{3\pi}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(2+2\rho+2\rho\ell)}$.

The reduced mass μ is defined using equation $\mu_i = \sqrt{m_i^2 - 2\mu^2\frac{dE(\omega,\mu,\rho)}{d\mu}}$ and $\mu_i = 2\mu_i$; then it is determined by Eq. (23).

$$\mu = \frac{1}{2}\left(m_i^2 + \frac{\Gamma(2+\rho+2\rho\ell)}{4\mu\rho^2\Gamma(3\rho+2\rho\ell)}\right)^{1/2} \quad (23)$$

The variation parameter ρ is a different value and depends on the potential type. For the ground and excited states, it can be defined by minimizing conditions of energy that are approximately defined by reducing the bound state oscillator frequency. Using the digamma function $\left(\Psi(x) = \frac{d(ln\Gamma(x))}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}\right)$ and condition of minimum frequency, we obtain the equation for determining the variation parameter ρ from the equation:

$$\frac{2}{\rho} - (2 + 2\ell)\Psi(2\rho + 2\rho\ell) + (1 + 2\rho\ell)\Psi(2 + \rho + 2\rho\ell) - (3 + 2\rho\ell)\Psi(3\rho + 2\rho\ell) + (4 + 2\rho\ell)\Psi(4\rho + 2\rho\ell) = 0 \quad (24)$$

Based on $\rho = \underset{\rho}{min}\,\omega(\rho,\ell)$, which is determined by Eq. (24):

ℓ	0	1	2	3	4
ρ	0.7178	0.7134	0.7116	0.7106	0.7099

Then, from Eq. (21), $E = E_r + iE_i$, the real (E_r), and the imaginary (E_i) energy eigenvalues in the d-dimensional auxiliary symplectic space R^d are defined. So, the imaginary and real parts of the energy eigenvalue in the ground state become:

$$E_r = -\frac{4\mu\rho^2 g^2\Gamma(2\rho)}{9\pi\,\Gamma(3\rho)\Gamma(2+2\rho)} + \left[\frac{m_D^2}{2} - \frac{4\pi}{g^2}\sigma\right]\frac{\Gamma(4\rho)\Gamma(2+2\rho)}{\Gamma(3\rho)\Gamma(2\rho)} \quad (25)$$

$$E_i = \frac{\pi T}{4}\left[\frac{g^2}{4} - \frac{\sigma\pi}{m_D^2}\right]\left(\frac{3\pi m_D}{2\mu\rho^2 g^2}\frac{\Gamma(4\rho)\Gamma(2+2\rho)}{\Gamma(3\rho)\Gamma(2\rho)} - 1\right) \quad (26)$$

Then, the mass spectra and the constituent mass of the charm can be calculated and defined as a finite lifetime and decay width at a finite temperature in an isotropic thermal medium. Hence, we utilized the relativistic mass contribution to calculate the bound state energy eigenvalue.

In Section 4, we used the transformation of two Symplectic intertwined spaces and defined all operators in the form of the normal ordering techniques. We defined the energy eigenvalue without the relativistic mass contribution. Now, using the definition and equations in Section 2, using the Feynman path integral method and the quantum field theory of high temperature environment, one can define the approximation form of $\sqrt{p^2 + m_0^2} \cong \frac{1}{2}\underset{\mu_i}{min}\left(\mu_i + \frac{p^2+m_0^2}{\mu_i}\right)$ for the creation of bound states. This approach gives us the solutions to the quantized energy values within the real and imaginary potential components of interaction between particle-antiparticle or particle-particle systems at finite temperatures. As described and explained in the above section, the decay width (ionization rate or transition probability) of an excited electron in the atom or dissipation rate for hadronic states based on a complex potential function can be calculated. So, the finite lifetime and the decay width were calculated and are presented in Table 1 for the ground and excited states at finite temperature with the critical temperature $T_c \approx 220\ MeV$.

The experimental data is taken from [27], and the theoretical data are taken from [28,29].

A brief explanation of real and imaginary parts of Eq. (21) is given below. As we presented above, the charmonium bound state energies and potential interaction have the complex forms $U = U_r + iU_i$ and $E = E_r + iE_i$ in quantum mechanics. The imaginary part of the potential U_i: (1) dominates at large distances where a larger U_i leads to a larger $|E_i|$ and a shorter lifetime τ, (2) causes the bound state to decay more rapidly, and (3) determines the stability of the bound state and the binding energy. Hence, it describes the dissolution of unstable bound states like charmonium and bottonium at high temperatures $T > 2.07T_c$. The imaginary part E_i is related to the finite lifetime τ of unstable states and the decay width by relation,

Table 1. Mass spectrum of charmonium bound state, the constituent mass of charm quark at finite temperature in the unit of GeV, the finite lifetime ($\tau = \frac{\hbar}{E_i}$) in the unit $10^{-10}s$, and the decay width ($\Gamma = |-2E_i|$) in the unit of GeV.

	T	0.01	0.02	0.06	0.13	0.16	0.19	0.20	$M_{teo.}$ [28,29]	$M_{exp.}$ [27]
$l = 0$	M ($M_c = 2.845$)	3.082	3.078	3.062	3.033	3.020	3.007	3.003	3.096 3.096	3.096
	μ_i	1.433	1.433	1.433	1.433	1.433	1.433	1.433	-	-
	τ	3.975	3.725	3.350	3.280	3.347	3.460	-	-	-
	Γ	0.806	0.860	0.957	0.977	0.958	0.926	-	-	-
$l = 1$	M ($M_c = 2.975$)	3.348	3.344	3.340	3.301	3.289	3.277	3.272	3.255 3.521	3.525
	μ_i	1.426	1.426	1.426	1.426	1.426	1.426	1.426	-	-
	τ	3.898	3.629	3.468	3.099	3.143	3.230	-	-	-
	Γ	0.822	0.883	0.924	1.034	1.020	0.992	-	-	-

$$\Gamma = \int dr\left|\Psi_n(r)\right|^2 U_i = = \int dr\left|\Psi_n(r)\right|^2 \Xi(m_D T) = -\,2E_i \quad (27)$$

This equation describes the rate at which the state decays in a bound state. The imaginary part of energy eigenvalue E_i for the n-th states, the bound state wavefunction Ψ_n is defined by the calculation of the expectation value $E_i = \langle\Psi_n|i\Gamma|\Psi_n\rangle$. The real part E_r corresponds to the binding energy of a bound state. A larger value of $|E_i|$ corresponds to a shorter lifetime and larger decay width. E_i in the long-distance behavior of a system in the thermal isotropic medium is the real part of the energy eigenvalue, which is reduced to the Coulomb-Yukawa-like potential energy eigenvalue. At short distances, the behavior of the confined state in the thermal isotropic medium is perturbative to the zero or vacuum state, and using this condition, we can determine a resonance state of the hadronic bound state and the width for a resonance state. Contrary to short and large distances, the interaction behavior is challenging to explain at the intermediate distance. In the intermediate range, where $rm_D \approx 1$, the interaction does not appear more straightforward compared to the asymptotic limits, and we do not present it here (for more details, see [21]). For the running interaction in Eq. (20), we define the temperature dependence of the Deby mass and string coefficient in the symplectic intertwined spaces R^d and including the lower temperatures close to $T_c \approx 220\ MeV$. Due to the importance of the dimensionless ratio

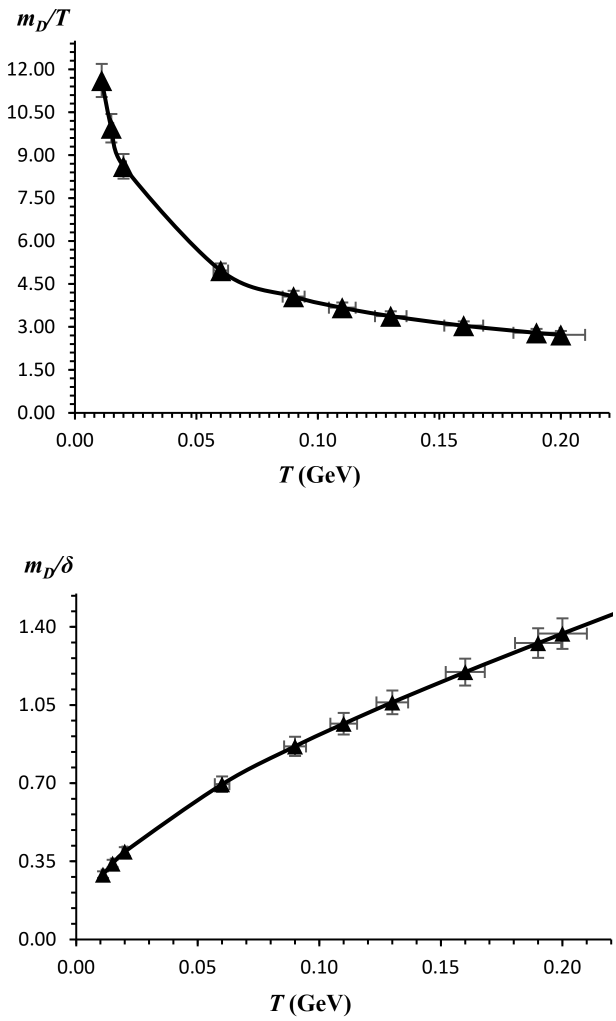

Fig. 1. The dimensionless ratio $\frac{m_D}{T}$ (up) and $\frac{m_D}{\delta}$ as a function of temperature (down) for the charmonium confined state in the ground state ($\ell = 0$) close to $T_c \approx 220\ MeV$. The experimental mass value of charmonium is 3.096 MeV, $\ell = 0$, and 3.525 MeV, $\ell = 1$.

$m_D\left[-\frac{g^2}{8\pi}m_D^2+\sigma\right]^{-0.5}=\frac{m_D}{\delta}$ in phenomenological modeling of the hadronic bound state, we plot those ratios versus temperature for the ground state in Fig. 1.

As it is commonly understood and widely known, the imaginary part of the potential interaction U_i for a bound state at finite temperature can have a significant impact in describing various phenomena at high energy physics. Mainly, it is associated with the effects of decay and dissociation of the system in a thermal medium. Hence, the constituent particle of bound states can interact with other quarks, antiquarks, and gluons in the thermal environment, leading to deconfining processes. Interactions with the surrounding particles can be described by imaginary components to the potential and energy eigenvalue as presented in Eqs. (26) and (27), which are related to the finite lifetime of the charmonium state due to its interactions with the medium. The description of U_i and E_i, including the effects of decay and dissociation, requires the specific advanced theoretical framework and computational methods, such as effective field theories and lattice quantum-chromodynamics approaches.

Calculation of these imaginary parts is important in high energy hadron physics. It is an active area of research in particle physics, and ongoing studies continue to refine our understanding of strong interactions and exotic bound states. Using the above equations, the variation of the real E_r and imaginary E_i energy eigenvalues, and also the imaginary term of potential interaction U_i for the charmonium bound state in the ground state ($\ell=0$) and the first excited ($\ell=1$) states in the isotropic medium at a finite temperature close to $T_c\approx 220\ MeV$ are determined and plotted in Fig. 2.

According to Fig. 3 and based on the theoretical results of interaction in the thermal isotropic medium presented in the symplectic d-dimensional space, the negative potential regions indicate the separation of two particles. In other words, in this region, the bound states will be on the threshold of overcoming the confirming potential so that a decrease in the absolute value of U_i will result in the charmonium bound states being no longer bound ($M \prec M_c=\sum_{i=1}^{n}\mu_i$), i.e., $T \succcurlyeq 300MeV$.

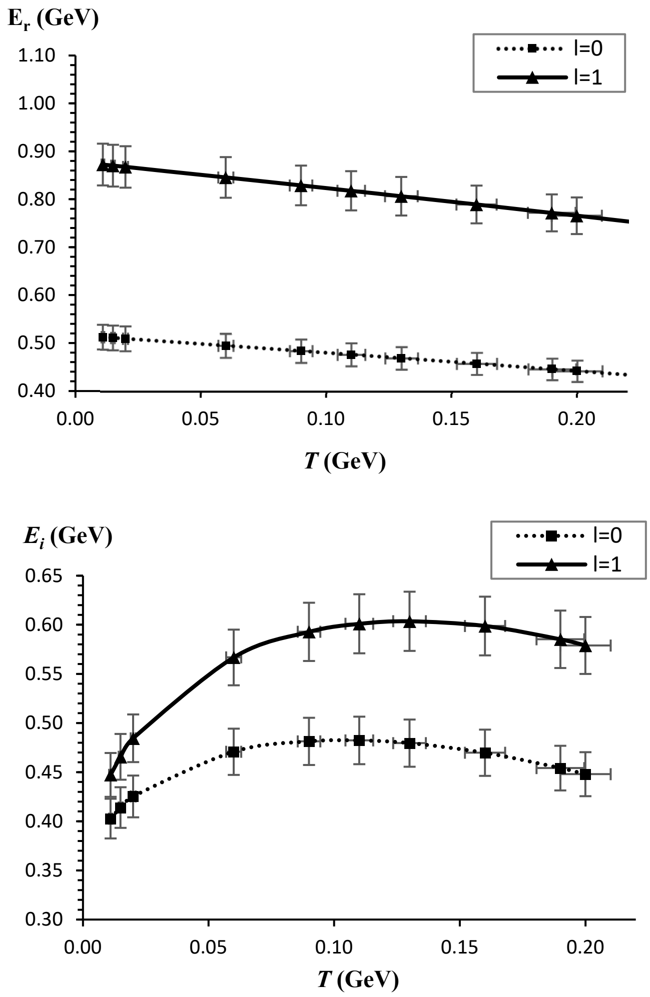

Fig. 2. The real E_r and imaginary E_i part of the energy eigenvalues for the ground ℓ ($\ell=0$) and the first excited ($\ell=1$) states versus temperature in the isotropic medium at a finite temperature close to $T_c\approx 220\ MeV$. The experimental mass value of charmonium is $3.096\ MeV$, $\ell=0$, and $3.525\ MeV$, $\ell=1$.

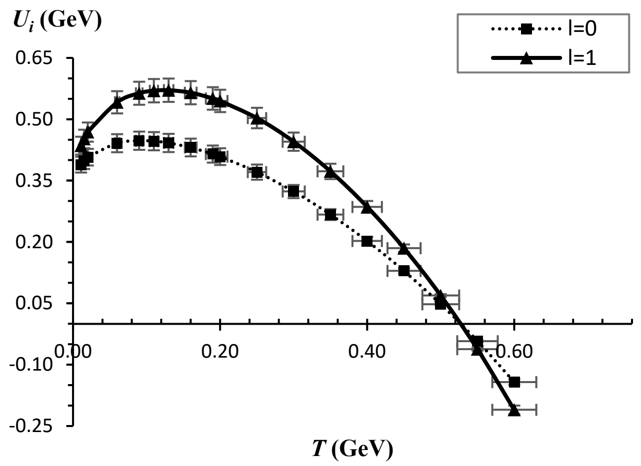

Fig. 3. The imaginary U_i part of potential interaction for the ground ($\ell=0$) and the first excited ($\ell=1$) states versus temperature in the isotropic medium at a finite temperature close to $T_c\approx 220\ MeV$. Experimental mass value of charmonium is $3.096\ MeV$, $\ell=0$, and $3.525\ MeV$, $\ell=1$.

5. Conclusion

The purpose of this theoretical research has been to present an analytical calculation of the real and imaginary potential of interaction with the Debye screened parameter of a charmonium bound states in a thermal medium at finite temperature based on the relativistic behavior of interaction within the Sturmian representation for two intertwined spaces and the mathematical method of quantum field theory techniques. The introduced potential estimates a certain real-time propagator function $G(\vec{r},t) = G(\vec{r},i\tau)$ for current $J(r) = R^{+}(r)R^{-}(r)$, whose Fourier transform determines the bound state properties and behavior. We modified this potential in the d-dimensional with canonical variable $r = q^{2\rho}$, then extracted from the correlator interactions based on functional dependence on τ and the Feynman path integral method. The characteristics of the confined state, such as mass spectrum, energy eigenvalue, and the relativistic correction on the constituent mass of charm quark at a finite temperature depending on Deby mass were determined using the nonperturbative and perturbative Hamiltonian in the relativistic limit. Apart from the standard terms of potential, we extracted the relativistic correction $H_{rel} = U_{rel}(r)\left[\left(1 + \frac{\ell(\ell+1)}{r^2c^2}\right)^{-0.5} - 1\right]$ to the total Hamiltonian. The mass spectra of charmonium for ground and excited states were determined using Eqs. (7) and (21). The free parameter μ was obtained by

$$\mu = \left(\frac{m_1^2+3\left[\frac{m_D^2}{2}-\frac{4\pi}{g^2}\sigma\right]\Gamma(4\rho+2\rho\ell)\Gamma(2+\rho+2\rho\ell)}{4-\frac{32(g^2\rho\Gamma(2\rho+2\rho\ell))^2}{9(4\pi)^2\Gamma(3\rho+2\rho\ell)\Gamma(2+\rho+2\rho\ell)}}\right)^{0.5}$$

and then the relativistic mass corrections μ_{rel} revealed itself in Eqs. (25) and (26) by using $\mu_{rel} = 2\mu$. The dependence of real and imaginary parts of energy eigenvalue on relativistic mass was obtained as $E_r \sim \mu$, $E_i \sim \frac{1}{\mu}$. Therefore, using the fitted values for parameters, we found the eigenenergy and mass spectrum of charmonium bound state for the ground and first excited state. Calculating the imaginary part of energy and potential gives us an explanation of the phenomenological properties of a system; so for charmonium systems, we adopted the numerical values of the critical mass, which is defined by the relativistic mass of constituent particles in the bound state for the ground and first excited state $M_c = 2.845\ MeV$ and $M_c = 2.975\ MeV$, respectively. It should be noted that the calculation of relativistic correction on the mass spectra of charmonium confined state in the thermal isotropic medium under finite temperature is in good agreement with experimental data. The values obtained are also in good agreement with the theoretical work of researchers. We defined the present results to compute the charmonium bound state mass for different principal quantum numbers and discussed the behavior of the real and imaginary terms of energy eigenvalue and the results obtained graphically. The exponential type potential, $U(r) = -\frac{g^2}{4\pi}\left[m_D + \frac{e^{-m_D r}}{r}\right] + 2\sigma\left[\frac{1}{m_D} + \frac{e^{-m_D r}-1}{r m_D^2}\right] + i\Xi(m_D T)$ has been successfully applied in predicting and approximating the mass spectra of charmonium confined states. The proposed analytical solutions were used to explain and describe other characteristics and thermal properties, such as the finite lifetime and the decay width in the hot medium under finite temperature.

Conflicts of interest

No potential conflict of interest was reported by the authors.

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Appendix A

$\Xi(m_D T)$ is the imaginary part of the potential function that describes the damping of the in-isotropic medium charmonium bound state propagator at long distances with σ coupling constant [21] and at short distances with α coupling constant [10]. The recognized Fourier transform pull-out constants were used to evaluate simple integral combined constants for the final result. Here is the solution of integrals in Eq. (9) ($a = \frac{p}{m_D}$).

$$\Xi(m_D r) = \Xi_{short}(m_D r) + \Xi_{long}(m_D T) = -\frac{g^2}{4\pi} T\theta(m_D T) + \frac{2\sigma T}{m_D^2}\,\xi(m_D T) =$$

$$-\frac{g^2}{2\pi} T \int_0^\infty \frac{a da}{(a^2+1)^2}\left[1 - \frac{\sin(a m_D r)}{a m_D r}\right] + \frac{4\sigma T}{m_D^2}\int_0^\infty \frac{da}{a(a^2+1)^2}\left[1 - \frac{\sin(a m_D r)}{a m_D r}\right] =$$

$$-\frac{g^2}{2\pi} T \int_0^\infty \frac{a, da}{(a^2+1)^2}\mathcal{F}\left[\frac{1}{(a^2+1)^2}\right] + \frac{4\sigma T}{m_D^2}\int_0^\infty \frac{da}{a(a^2+1)^2}\mathcal{F}\left[\frac{1}{a(a^2+1)^2}\right] =$$

$$-\frac{g^2}{2\pi} T\left[\int_0^\infty \frac{a, da}{(a^2+1)^2}\left(\frac{\pi}{2} e^{-|m_D r|}\right) = \left(\frac{\pi}{2} e^{-|m_D r|}\right)\int_0^\infty \frac{a, da}{(a^2+1)^2} = \left(\frac{\pi}{2} e^{-|m_D r|}\right)\left(\frac{\pi}{4}\right)\right] + \frac{4\sigma T}{m_D^2}\left[\int_0^\infty \frac{da}{a(a^2+1)^2}\left(\frac{\pi}{4} e^{-|m_D r|}\right) =\right.$$

$$\left.\left(\frac{\pi}{4} e^{-|m_D r|}\right)\int_0^\infty \frac{da}{a(a^2+1)^2} = \left(\frac{\pi}{4} e^{-|m_D r|}\right)\left(\frac{\pi}{4}\right)\right] =$$

$$-\frac{g^2}{2\pi} T\,\frac{\pi^2}{8} e^{-|m_D r|} + \frac{4\sigma T}{m_D^2}\frac{\pi^2}{16} e^{-|m_D r|} \tag{A.1}$$